Numerical Analysis and Computing

Lecture Notes #08
— Numerical Differentiation and Integration —
Composite Numerical Integration; Romberg Integration
Adaptive Quadrature / Gaussian Quadrature

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Outline

Divide and Conquer with Simpson's Rule I/III

The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

Simpson's Rule with $h = 2$

$$\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

The error is **−3.17143** (5.92%).

Divide and Conquer with Simpson's Rule I/III

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$$\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

The error is $-$**3.17143** (5.92%).

Divide-and-Conquer: Simpson's Rule with $h = 1$

$$\int_0^2 e^x dx + \int_2^4 e^x dx \approx \frac{1}{3}(e^0 + 4e^1 + e^2) + \frac{1}{3}(e^2 + 4e^3 + e^4) = 53.86385$$

The error is $-$**0.26570**. (0.50%) Improvement by a factor of 10!

Divide and Conquer with Simpson's Rule II/III

The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

Divide-and-Conquer: Simpson's Rule with $h = 1/2$

$$\int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 e^x dx \approx \frac{1}{6}(e^0 + 4e^{1/2} + e^1) + \frac{1}{6}(e^1 + 4e^{3/2} + e^2) + \frac{1}{6}(e^2 + 4e^{5/2} + e^3) + \frac{1}{6}(e^3 + 4e^{7/2} + e^4) = 53.61622$$

The error has been reduced to $-$**0.01807** (0.034%).

Divide and Conquer with Simpson's Rule II/III

The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

Divide-and-Conquer: Simpson's Rule with $h = 1/2$

$$\int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 e^x dx \approx \frac{1}{6}(e^0 + 4e^{1/2} + e^1) + \frac{1}{6}(e^1 + 4e^{3/2} + e^2)$$
$$+\frac{1}{6}(e^2 + 4e^{5/2} + e^3) + \frac{1}{6}(e^3 + 4e^{7/2} + e^4) = 53.61622$$

The error has been reduced to **-0.01807** (0.034%).

h	abs-error	err/h	err/h^2	err/h^3	err/h^4
2	3.17143	1.585715	0.792857	0.396429	0.198214
1	0.26570	0.265700	0.265700	0.265700	0.265700
1/2	0.01807	0.036140	0.072280	0.144560	0.289120

Divide and Conquer with Simpson's Rule III/III

Extending the table...

h	abs-error	err/h	err/h^2	err/h^3	**err/h^4**	err/h^5
2	3.171433	1.585716	0.792858	0.396429	**0.198215**	0.099107
1	0.265696	0.265696	0.265696	0.265696	**0.265696**	0.265696
1/2	0.018071	0.036142	0.072283	0.144566	**0.289132**	0.578264
1/4	0.001155	0.004618	0.018473	0.073892	**0.295566**	1.182266
1/8	0.000073	0.000580	0.004644	0.037152	**0.297215**	2.377716

Clearly, the **err/h^4** column seems to converge (to a non-zero constant) as $h \searrow 0$. The columns to the left seem to converge to zero, and the err/h^5 column seems to grow.

This is **numerical evidence** that the composite Simpson's rule has a convergence rate of $\mathcal{O}\left(h^4\right)$. *But, isn't Simpson's rule 5th order???*

Generalized Composite Simpson's Rule I/II

For an even integer n: Subdivide the interval $[a, b]$ into n subintervals, and apply Simpson's rule on each consecutive pair of sub-intervals. With $h = (b-a)/n$ and $x_j = a + jh$, $j = 0, 1, \ldots, n$, we have

$$\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3} \left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) \right] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\},$$

for some $\xi_j \in [x_{2j-2}, x_{2j}]$, if $f \in C^4[a, b]$.

Since all the interior "even" x_{2j} points appear twice in the sum, we can simplify the expression a bit...

Generalized Composite Simpson's Rule II/II

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) - f(x_n) + \sum_{j=1}^{n/2}\left[4f(x_{2j-1}) + 2f(x_{2j})\right]\right] - \frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j).$$

The error term is:

$$E(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j), \quad \xi_j \in [x_{2j-2}, x_{2j}]$$

The Error for Composite Simpson's Rule I/II

If $f \in C^4[a,b]$, the **Extreme Value Theorem** implies that $f^{(4)}$ assumes its max and min in $[a,b]$. Now, since

$$\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x),$$

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$$\left[\frac{n}{2}\right] \min_{x\in[a,b]} f^{(4)}(x) \leq \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \left[\frac{n}{2}\right] \max_{x\in[a,b]} f^{(4)}(x),$$

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$$\min_{x\in[a,b]} f^{(4)}(x) \leq \left[\frac{2}{n}\right] \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x),$$

The Error for Composite Simpson's Rule I/II

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$$\min_{x\in[a,b]} f^{(4)}(x) \leq f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x),$$

$$\left[\frac{n}{2}\right] \min_{x\in[a,b]} f^{(4)}(x) \leq \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \left[\frac{n}{2}\right] \max_{x\in[a,b]} f^{(4)}(x),$$

$$\min_{x\in[a,b]} f^{(4)}(x) \leq \left[\frac{2}{n}\right] \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x),$$

By the **Intermediate Value Theorem** $\exists \mu \in (a,b)$ so that

$$f^{(4)}(\mu) = \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \quad \Leftrightarrow \quad \frac{n}{2} f^{(4)}(\mu) = \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

The Error for Composite Simpson's Rule II/II

We can now rewrite the error term:

$$E(f) = -\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{h^5}{180} n f^{(4)}(\mu),$$

or, since $h = (b-a)/n \Leftrightarrow n = (b-a)/h$, we can write

$$E(f) = -\frac{(b-a)}{180} \mathbf{h^4 f^{(4)}}(\mu).$$

Hence **Composite Simpson's Rule** has **degree of accuracy 3** (since it is exact for polynomials up to order 3), and the error is proportional to $\mathbf{h^4}$ — **Convergence Rate** $\mathcal{O}\left(h^4\right)$.

Composite Simpson's Rule — Algorithm

Algorithm (Composite Simpson's Rule)

Given the end points a and b and an even positive integer n:

[1] $h = (b - a)/n$

[2] `ENDPTS = f(a)+f(b)`
`ODDPTS = 0`
`EVENPTS = 0`

[3] FOR $i = 1, \ldots, n-1$ — (interior points)
$x = a + i * h$
if i is even: `EVENPTS += f(x)`
if i is odd: `ODDPTS += f(x)`
END

[4] `INTAPPROX = h*(ENDPTS+2*EVENPTS+4*ODDPTS)/3`

Homework #7 — Due Friday 11/13/2009, 12-noon

(Part-1)

Implement Composite Simpson's Rule, and use your code to solve `BF-4.4.3-a,b,c,d.`

Romberg Integration

The Return of Richardson's Extrapolation

Romberg Integration is the combination of the **Composite Trapezoidal Rule** (CTR)

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j)\right] - \frac{(b-a)}{12}h^2 f''(\mu)$$

and **Richardson Extrapolation**.

Here, we know that the error term for regular Trapezoidal Rule is $\mathcal{O}(h^3)$. By the same argument as for Composite Simpson's Rule, this gets reduced to $\mathcal{O}(\mathbf{h^2})$ for the composite version.

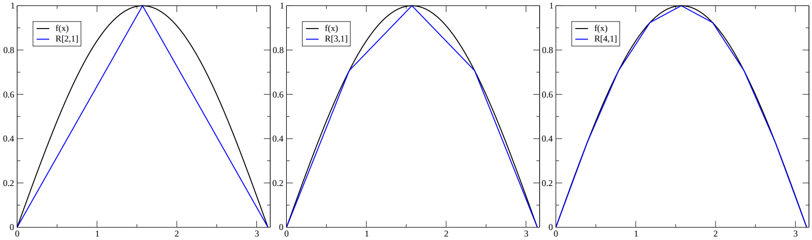
Romberg Integration

Step-1: CTR Refinement

Let $R_{k,1}$ denote the Composite Trapezoidal Rule with 2^{k-1} sub-intervals, and $h_k = (b-a)/2^{k-1}$. We get:

$$\begin{aligned} R_{1,1} &= \frac{h_1}{2}\left[f(a) + f(b)\right] \\ R_{2,1} &= \frac{h_2}{2}\left[f(a) + 2f(a+h_2) + f(b)\right] \\ &= \frac{(b-a)}{4}\left[f(a) + f(b) + 2f(a+h_2)\right] \\ &= \frac{1}{2}\left[R_{1,1} + h_1 f(a+h_2)\right] \\ &\vdots \\ R_{k,1} &= \underbrace{\frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(2i-1)h_k)\right]}_{\text{Update formula, using previous value + new points}} \end{aligned}$$

Example: $R_{k,1}$ for $\int_0^{\pi} \sin(x)dx$



k	$R_{k,1}$
1	0
2	1.5707963267949
3	1.8961188979370
4	1.9742316019455
5	1.9935703437723
6	1.9983933609701
7	1.9995983886400

Extrapolate using Richardson

We know that the error term is $\mathcal{O}(h^2)$, so in order to eliminate this term we combine to consecutive entries $R_{k-1,1}$ and $R_{k,1}$ to form a higher order approximation $R_{k,2}$ of the integral.

$$\mathbf{R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{2^2 - 1}}$$

$R_{k,1}$ — $\mathcal{O}(h^2)$	$R_{k,2}$
0	0
1.5707963267949	2.09439510239
1.8961188979370	2.00455975498
1.9742316019455	2.00026916994
1.9935703437723	2.00001659104
1.9983933609701	2.00000103336
1.9995983886400	2.00000006453

Extrapolate, again...

It turns out (Taylor expand to check) that the complete error term for the Trapezoidal rule only has even powers of h:

$$\int_a^b f(x) = R_{k,1} - \sum_{i=1}^{\infty} E_{2i} h_k^{2i}.$$

Hence the $R_{k,2}$ approximations have error terms that are of size $\mathcal{O}(\mathbf{h^4})$.

To get $\mathcal{O}(h^6)$ approximations, we compute

$$\mathbf{R_{k,3} = R_{k,2} + \frac{R_{k,2} - R_{k-1,2}}{4^2 - 1}}$$

Extrapolate, yet again...

In general, since we only have even powers of h in the error expansion:

$$\mathbf{R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}}$$

Revisiting $\int_0^{\pi} \sin(x)dx$:

$R_{k,1}$ — $\mathcal{O}\left(h^2\right)$	$R_{k,2}$ — $\mathcal{O}\left(h^4\right)$	$R_{k,3}$ — $\mathcal{O}\left(h^6\right)$	$R_{k,4}$ — $\mathcal{O}\left(h^8\right)$
0			
1.570796326794897	2.094395102393195		
1.896118897937040	2.004559754984421	1.998570731823836	
1.974231601945551	2.000269169948388	1.999983130945986	2.000005549979671
1.993570343772340	2.000016591047935	1.999999752454572	2.000000016288042
1.998393360970145	2.000001033369413	1.999999996190845	2.000000000059674
1.999598388640037	2.000000064530001	1.999999999940707	2.000000000000229

Homework? No, enough already — Here's the code outline!

Code (Romberg Quadrature)

```
% Romberg Integration for sin(x) over [0,pi]
a = 0; b = pi; % The Endpoints
R = zeros(7,7);
R(1,1) = (b-a)/2*(sin(a)+sin(b));
for k=2:7
  h = (b-a)/2^(k-1);
  R(k,1)=1/2*(R(k-1,1)+2*h*sum(sin(a+(2*(1:(2^(k-2)))-1)*h)));
end
for j=2:7
  for k=j:7
   R(k,j) = R(k,j-1)+(R(k,j-1)-R(k-1,j-1))/(4^(j-1)-1);
  end
end

disp(R)
```

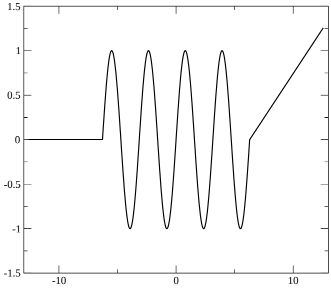
More Advanced Numerical Integration Ideas

Adaptive and Gaussian Quadrature

Introduction

The **composite formulas** require **equally spaced nodes**.

This is not good if the function we are trying to integrate has both regions with large fluctuations, and regions with small variations.



We need many points where the function fluctuates, but few points where it is close to constant or linear.

Introduction — Adaptive Quadrature Methods

Idea Cleverly predict (or measure) the amount of variation and automatically add more points where needed.

We are going to discuss this in the context of Composite Simpson's rule, but the approach can be adopted for other integration schemes.

First we are going to develop a way to **measure the error** — a numerical estimate of the actual error in the numerical integration. Note: just knowing the structure of the error term is not enough! (We will however use the structure of the error term in our derivation of the numerical error estimate.)

Then we will use the error estimate to decide whether to accept the value from CSR, or if we need to refine further (recompute with smaller h).

Some Notation — One-step Simpson's Rule $S(f; a, b)$

Notation — "One-step" Simpson's Rule:

$$\int_a^b f(x)\,dx = S(f;\, a, b) - \underbrace{\frac{h_1^5}{90} f^{(4)}(\mu_1)}_{\mathbf{E(f;\, h_1, \mu_1)}}, \quad \mu_1 \in (a, b),$$

where

$$S(f;\, a, b) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right], \quad h_1 = \frac{(b-a)}{2}.$$

Composite Simpson's Rule (CSR)

With this notation, we can write CSR with $n = 4$, and $h_2 = (b-a)/4 = h_1/2$:

$$\int_a^b f(x)\,dx = S(f;\, a, \tfrac{a+b}{2}) + S(f\,;\tfrac{a+b}{2}, b) - E(f; h_2, \mu_2).$$

We can squeeze out an estimate for the error by noticing that

$$E(f; h_2, \mu_2) = \frac{1}{16}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2)\right) = \frac{1}{16} E(f; h_1, \mu_2).$$

Now, **assuming** $f^{(4)}(\mu_1) \approx f^{(4)}(\mu_2)$, we do a little bit of algebra magic with our two approximations to the integral...

Wait! Wait! Wait! — I pulled a fast one!

$$E(f; h_2, \mu_2) = \frac{1}{32}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2^1)\right) + \frac{1}{32}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2^2)\right)$$

where $\mu_2^1 \in [a, \frac{a+b}{2}]$, $\mu_2^2 \in [\frac{a+b}{2}, b]$.

If $f \in C^4[a, b]$, then we can use our old friend, the **intermediate value theorem**:

$$\exists \mu_2 \in [\mu_2^1, \mu_2^2] \subset [a, b] : \ f^{(4)}(\mu_2) = \frac{f^{(4)}(\mu_2^1) + f^{(4)}(\mu_2^2)}{2}.$$

So it follows that

$$E(f; h_2, \mu_2) = \frac{1}{16}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2)\right).$$

Back to the Error Estimate...

Now we have

$$S(f;\, a, \tfrac{a+b}{2}) + S(f;\, \tfrac{a+b}{2}, b) - \frac{1}{16}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2)\right)$$
$$= S(f;\, a, b) - \frac{h_1^5}{90} f^{(4)}(\mu_1).$$

Now use the assumption $f^{(4)}(\mu_1) \approx f^{(4)}(\mu_2)$ (and replace μ_1 and μ_2 by μ):

$$\mathbf{\frac{h_1^5}{90} f^{(4)}(\mu)} \approx \frac{16}{15}\Big[S(f;\, a, b) - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b)\Big],$$

notice that $\frac{h_1^5}{90} f^{(4)}(\mu) = E(f;\, h_1, \mu) = 16E(f;\, h_2, \mu)$. Hence

$$E(f;\, h_2, \mu) \approx \frac{1}{15}\Big[S(f;\, a, b) - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b)\Big].$$

Finally, we have the error estimate in hand...

Using the estimate of $\frac{h_1^5}{90} f^{(4)}(\mu)$, we have

Error Estimate for CSR

$$\left| \int_a^b f(x)dx - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b) \right| \approx \frac{1}{15} \left| S(f;\, a, b) - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b) \right|$$

Notice!!! $S(f;\, a, (a+b)/2) + S(f;\, (a+b)/2, b)$ approximates $\int_a^b f(x)dx$ **15 times better** than it agrees with the known quantity $S(f;\, a, b)$!!!

Example — Error Estimates I/II

We will apply Simpson's rule to

$$\int_0^{\pi/2} \sin(x)\,dx = 1.$$

Here,

$$\mathbb{S}_1(\sin(x);\ 0, \pi/2) = S(\sin(x);\ 0, \pi/2)$$

$$= \frac{\pi}{12}\left[\sin(0) + 4\sin(\pi/4) + \sin(\pi/2)\right] = \frac{\pi}{12}\left[2\sqrt{2} + 1\right]$$

$$= 1.00227987749221.$$

$$\mathbb{S}_2(\sin(x);\ 0, \pi/2) = S(\sin(x);\ 0, \pi/4) + S(\sin(x);\ \pi/4, \pi/2)$$

$$= \frac{\pi}{24}\left[\sin(0) + 4\sin(\pi/8) + 2\sin(\pi/4) + 4\sin(3\pi/8) + \sin(\pi/2)\right]$$

$$= 1.00013458497419.$$

Example — Error Estimates II/II

The error estimate is given by

$$\begin{aligned}
&\frac{1}{15}\left[\mathbb{S}_1(\sin(x);\ 0, \pi/2) - \mathbb{S}_2(\sin(x);\ 0, \pi/2)\right] \\
&\quad = \frac{1}{15}\left[1.00227987749221 - 1.00013458497419\right] \\
&\quad = 0.00014301950120.
\end{aligned}$$

This is a very good approximation of the actual error, which is 0.00013458497419.

OK, we know how to get an error estimate. How do we use this to create an adaptive integration scheme???

Adaptive Quadrature

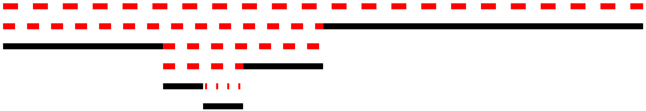
We want to approximate $\mathcal{I} = \int_a^b f(x)\,dx$ with an error less than ϵ (a specified tolerance).

[1] Compute the two approximations
$\mathbb{S}_1(f(x);\, a, b) = S(f(x);\, a, b)$, and
$\mathbb{S}_2(f(x);\, a, b) = S(f(x);\, a, \frac{a+b}{2}) + S(f(x);\, \frac{a+b}{2}, b)$.

[2] Estimate the error, if the estimate is less than ϵ, we are done. Otherwise...

[3] Apply steps [1] and [2] recursively to the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ with tolerance $\epsilon/2$.

Adaptive Quadrature, Interval Refinement Example #1



The funny figure above is supposed to illustrate a possible sub-interval refinement hierarchy. **Red** dashed lines illustrate failure to satisfy the tolerance, and **black** lines illustrate satisfied tolerance.

level	tol	interval
1	ϵ	$[a, b]$
2	$\epsilon/2$	$[a, a+\frac{b-a}{2}]$ $[a+(b-a)/2, b]$
3	$\epsilon/4$	$[a, a+\frac{b-a}{4}]$ $[a+\frac{b-a}{4}, a+\frac{b-a}{2}]$
⋮		

Adaptive Quadrature, Interval Refinement Example #2

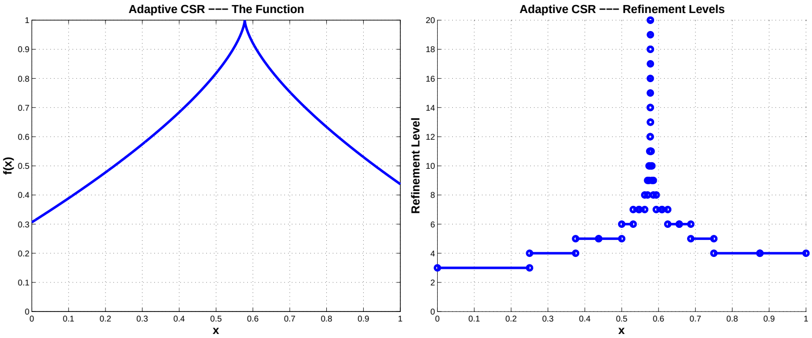


Figure: Application of adaptive CSR to the function $f(x) = 1 - \sqrt[3]{(x - \frac{\pi}{2e})^2}$. Here, we have required that the estimated error be less than 10^{-6}. The left panel shows the function, and the right panel shows the number of refinement levels needed to reach the desired accuracy. At completion we have the value of the integral being 0.61692712, with an estimated error of $3.93 \cdot 10^{-7}$.

Gaussian Quadrature

Idea: Evaluate the function at a set of **optimally chosen** points in the interval.

We will choose $\{x_0, x_1, \dots, x_n\} \in [a, b]$ and coefficients c_i, so that the approximation

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$$

is exact for the largest class of polynomials possible.

We have already seen that the open Newton-Cotes formulas sometimes give us better "bang-for-buck" than the closed formulas (*e.g.* the mid-point formula uses only 1 point and is as accurate as the two-point trapezoidal rule). — Gaussian quadrature takes this one step further.

Quadrature Types — A Comparison

	Newton-Cotes		Gaussian
	Open	Closed	
Quadrature Points	Degree of Accuracy	Degree of Accuracy	Degree of Accuracy
1	**1***	—	1
2	1	**1**†	3
3	3	**3**#	5
4	3	3	7
5	5	5	9

* — The mid-point rule.
† — Trapezoidal rule.
# — Simpson's rule.

The mid-point rule is the only optimal scheme we have see so far.

Gaussian Quadrature — Example

2-Point Formula

Suppose we want to find an optimal two-point formula:

$$\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2).$$

Since we have 4 parameters to play with, we can generate a formula that is **exact up to polynomials of degree 3**. We get the following 4 equations:

$$\begin{aligned}
\int_{-1}^{1} 1\,dx &= 2 = c_1 + c_2 \\
\int_{-1}^{1} x\,dx &= 0 = c_1 x_1 + c_2 x_2 \\
\int_{-1}^{1} x^2\,dx &= \tfrac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \\
\int_{-1}^{1} x^3\,dx &= 0 = c_1 x_1^3 + c_2 x_2^3
\end{aligned}
\qquad \Bigg\| \qquad
\begin{aligned}
c_1 &= 1 \\
c_2 &= 1 \\
x_1 &= -\frac{\sqrt{3}}{3} \\
x_2 &= \frac{\sqrt{3}}{3}
\end{aligned}$$

Higher Order Gaussian Quadrature Formulas

We could obtain higher order formulas by adding more points, computing the integrals, and solving the resulting non-linear system of equations... but it gets very painful, very fast.

The **Legendre Polynomials** come to our rescue!

The Legendre polynomials $P_n(x)$ are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$, *i.e.*

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \alpha_n \delta_{n,m} = \begin{cases} 0 & m \neq n \\ \alpha_n & m = n. \end{cases}$$

If $P(x)$ is a polynomial of degree less than n, then

$$\int_{-1}^{1} P_n(x)P(x)\,dx = 0.$$

A Quick Note on Legendre Polynomials

We will see Legendre polynomials in **more detail later**. For now, all we need to know is that they satisfy the property

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \alpha_n\delta_{n,m}.$$

and the first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - 1/3 \\
P_3(x) &= x^3 - 3x/5 \\
P_4(x) &= x^4 - 6x^2/7 + 3/35 \\
P_5(x) &= x^5 - 10x^3/9 + 5x/21.
\end{aligned}$$

It turns out that the **roots** of the Legendre polynomials are the nodes in Gaussian quadrature.

Higher Order Gaussian Quadrature Formulas

Theorem

Suppose that $\{x_1, x_2, \dots, x_n\}$ *are the roots of the* n^{th} *Legendre polynomial* $P_n(x)$ *and that for each* $i = 1, 2, \dots, n$, *the coefficients* c_i *are defined by*

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ *is any polynomial of degree less than* $2n$, *then*

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$

Proof of the Theorem I/III

Let us first consider a polynomial, $P(x)$ with degree less than n. $P(x)$ can be rewritten as an $(n-1)$-st Lagrange polynomial with nodes at the roots of the n^{th} Legendre polynomial $P_n(x)$. This representation is exact since the error term involves the n^{th} derivative of $P(x)$, which is zero. Hence,

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} \left[\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} P(x_j) \right] dx$$

$$= \sum_{i=1}^{n} \left[\int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\, dx \right] P(x_j) = \sum_{i=1}^{n} c_i P(x_i),$$

which verifies the result for polynomials of degree less than n.

Proof of the Theorem II/III

If the polynomial $P(x)$ of degree $[n, 2n)$ is divided by the n^{th} Legendre polynomial $P_n(x)$, we get:

$$P(x) = Q(x)P_n(x) + R(x)$$

where both $Q(x)$ and $R(x)$ are of degree less than n.

[1] Since $\deg(Q(x)) < n$

$$\int_{-1}^{1} Q(x)P_n(x)\,dx = 0.$$

[2] Further, since x_i is a root of $P_n(x)$:

$$P(x_i) = Q(x_i)P_n(x_i) + R(x_i) = R(x_i).$$

Proof of the Theorem III/III

[3] Now, since $\deg(R(x)) < n$, the first part of the proof implies

$$\int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i).$$

Putting [1], [2] and [3] together we arrive at

$$\begin{aligned}\int_{-1}^{1} P(x)\,dx &= \int_{-1}^{1} \left[Q(x)P_n(x) + R(x)\right]\,dx \\ &= \int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i) \\ &= \sum_{i=1}^{n} c_i P(x_i),\end{aligned}$$

which shows that the formula is exact for all polynomials $P(x)$ of degree less than $2n$. $\square$

Gaussian Quadrature beyond the interval $[-1, 1]$

By a simple linear transformation,

$$t = \frac{2x - a - b}{b - a} \quad \Leftrightarrow \quad x = \frac{(b-a)t + (b+a)}{2},$$

we can apply the Gaussian Quadrature formulas to any interval

$$\int_a^b f(x)\, dx = \int_{-1}^1 f\left(\frac{(b-a)t + (b+a)}{2}\right) \frac{(b-a)}{2} dt.$$

Examples I/II

Degree	$P_n(x)$	Roots / Quadrature points
2	$x^2 - 1/3$	$\{-1/\sqrt{3},\ 1/\sqrt{3}\}$
3	$x^3 - 3x/5$	$\{-\sqrt{3/5},\ 0,\ \sqrt{3/5}\}$
4	$x^4 - 6x^2/7 + 3/35$	$\{-0.86114,\ -0.33998,\ 0.33998,\ 0.86114\}$

Table: Quadrature points on “standard interval.

$$\int_0^{\pi/4} (\cos(x))^2\, dx = \frac{1}{4} + \frac{\pi}{8} = 0.642699081698724$$

Degree	Quadrature points	Coefficients
2	0.16597, 0.61942	1, 1
3	0.08851, 0.39270, 0.69688	0.55556, 0.88889, 0.55556
4	0.05453, 0.25919, 0.52621, 0.73087	0.34785, 0.65215, 0.65215, 0.34785

Table: Quadrature points translated to interval of interest; with weight coefficients.

Examples II/II

$$\int_0^{\pi/4} (\cos(x))^2 \, dx = \frac{1}{4} + \frac{\pi}{8} = 0.642699081698724$$

Degree	Quadrature points	Coefficients
2	0.16597, 0.61942	1, 1
3	0.08851, 0.39270, 0.69688	0.55556, 0.88889, 0.55556
4	0.05453, 0.25919, 0.52621, 0.73087	0.34785, 0.65215, 0.65215, 0.34785

Table: Quadrature points translated to interval of interest; with weight coefficients.

Degree	Integral approximation	Error
2	0.642317235049753	0.0003818466489...
3	0.642701112090729	0.0000020303920...
4	0.642699075999924	0.0000000056988...

Table: Approximation and Error, for GQ.