

Numerical Analysis and Computing

Lecture Notes #13

— Approximation Theory — Rational Function Approximation

Joe Mahaffy,
(mahaffy@math.sdsu.edu)

Department of Mathematics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

<http://www-rohan.sdsu.edu/~jmahaffy>

Spring 2010

Outline

- 1 **Approximation Theory**
 - Pros and Cons of Polynomial Approximation
 - New Bag-of-Tricks: Rational Approximation
 - Padé Approximation: Example #1

- 2 **Padé Approximation**
 - Example #2
 - Finding the Optimal Padé Approximation

Polynomial Approximation: Pros and Cons.

Advantages of Polynomial Approximation:

- [1] We can approximate any continuous function on a closed interval to within arbitrary tolerance. (*Weierstrass approximation theorem*)

Polynomial Approximation: Pros and Cons.

Advantages of Polynomial Approximation:

- [1] We can approximate any continuous function on a closed interval to within arbitrary tolerance. (*Weierstrass approximation theorem*)
- [2] Easily evaluated at arbitrary values. (*e.g. Horner's method*)

Polynomial Approximation: Pros and Cons.

Advantages of Polynomial Approximation:

- [1] We can approximate any continuous function on a closed interval to within arbitrary tolerance. (*Weierstrass approximation theorem*)
- [2] Easily evaluated at arbitrary values. (*e.g. Horner's method*)
- [3] Derivatives and integrals are easily determined.

Polynomial Approximation: Pros and Cons.

Advantages of Polynomial Approximation:

- [1] We can approximate any continuous function on a closed interval to within arbitrary tolerance. (*Weierstrass approximation theorem*)
- [2] Easily evaluated at arbitrary values. (*e.g. Horner's method*)
- [3] Derivatives and integrals are easily determined.

Disadvantage of Polynomial Approximation:

- [1] Polynomials tend to be oscillatory, which causes errors. This is sometimes, but not always, fixable: — *E.g.* if we are free to select the node points we can minimize the interpolation error (*Chebyshev polynomials*), or optimize for integration (*Gaussian Quadrature*).

Moving Beyond Polynomials: Rational Approximation

We are going to use rational functions, $r(x)$, of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^n p_i x^i}{1 + \sum_{j=1}^m q_j x^j}$$

and say that the degree of such a function is $N = n + m$.

Moving Beyond Polynomials: Rational Approximation

We are going to use rational functions, $r(x)$, of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^n p_i x^i}{1 + \sum_{j=1}^m q_j x^j}$$

and say that the degree of such a function is $N = n + m$.

Since this is a richer class of functions than polynomials — rational functions with $q(x) \equiv 1$ are polynomials, we expect that **rational approximation of degree N gives results that are at least as good as polynomial approximation of degree N .**

Padé Approximation

Extension of **Taylor expansion** to rational functions; selecting the p_i 's and q_i 's so that $r^{(k)}(x_0) = f^{(k)}(x_0) \forall k = 0, 1, \dots, N$.

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)}$$

Padé Approximation

Extension of **Taylor expansion** to rational functions; selecting the p_i 's and q_i 's so that $r^{(k)}(x_0) = f^{(k)}(x_0) \forall k = 0, 1, \dots, N$.

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)}$$

Now, use the Taylor expansion $f(x) \sim \sum_{i=0}^{\infty} a_i(x - x_0)^i$, for simplicity $x_0 = 0$:

$$f(x) - r(x) = \frac{\sum_{i=0}^{\infty} a_i x^i - \frac{\sum_{i=0}^m q_i x^i - \sum_{i=0}^n p_i x^i}{q(x)}}{q(x)}.$$

Padé Approximation

Extension of **Taylor expansion** to rational functions; selecting the p_i 's and q_i 's so that $r^{(k)}(x_0) = f^{(k)}(x_0) \forall k = 0, 1, \dots, N$.

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)}$$

Now, use the Taylor expansion $f(x) \sim \sum_{i=0}^{\infty} a_i(x - x_0)^i$, for simplicity $x_0 = 0$:

$$f(x) - r(x) = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^m q_i x^i - \sum_{i=0}^n p_i x^i}{q(x)}.$$

Next, we choose p_0, p_1, \dots, p_n and q_1, q_2, \dots, q_m so that the numerator has no terms of degree $\leq N$.

Padé Approximation: The Mechanics.

For simplicity we (sometimes) define the “indexing-out-of-bounds” coefficients:

$$\begin{cases} p_{n+1} = p_{n+2} = \cdots = p_N = 0 \\ q_{m+1} = q_{m+2} = \cdots = q_N = 0, \end{cases}$$

so we can express the **coefficients of x^k** in

$$\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^m q_i x^i - \sum_{i=0}^n p_i x^i = 0, \quad k = 0, 1, \dots, N$$

as

$$\sum_{i=0}^k a_i q_{k-i} = p_k, \quad k = 0, 1, \dots, N.$$

Padé Approximation: Abstract Example

1 of 2

Find the Padé approximation of $f(x)$ of degree 5, where $f(x) \sim a_0 + a_1x + \dots + a_5x^5$ is the Taylor expansion of $f(x)$ about the point $x_0 = 0$.

The corresponding equations are:

x^0	a_0	$-$	p_0	$=$	0
x^1	$a_0q_1 + a_1$	$-$	p_1	$=$	0
x^2	$a_0q_2 + a_1q_1 + a_2$	$-$	p_2	$=$	0
x^3	$a_0q_3 + a_1q_2 + a_2q_1 + a_3$	$-$	p_3	$=$	0
x^4	$a_0q_4 + a_1q_3 + a_2q_2 + a_3q_1 + a_4$	$-$	p_4	$=$	0
x^5	$a_0q_5 + a_1q_4 + a_2q_3 + a_3q_2 + a_4q_1 + a_5$	$-$	p_5	$=$	0

Note: $p_0 = a_0$!!! (This reduces the number of unknowns and equations by one (1).)

Padé Approximation: Abstract Example

2 of 2

We get a linear system for p_1, p_2, \dots, p_N and q_1, q_2, \dots, q_N :

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

If we want $n = 3$, $m = 2$:

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & \mathbf{a_0} & & \\ a_3 & a_2 & \mathbf{a_1} & \mathbf{a_0} & \\ a_4 & a_3 & \mathbf{a_2} & \mathbf{a_1} & \mathbf{a_0} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Padé Approximation: Abstract Example

2 of 2

We get a linear system for p_1, p_2, \dots, p_N and q_1, q_2, \dots, q_N :

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

If we want $n = 3$, $m = 2$:

$$\begin{bmatrix} a_0 & & -1 & & & \\ a_1 & a_0 & & & & \\ a_2 & a_1 & 0 & & & \\ a_3 & a_2 & 0 & a_0 & & \\ a_4 & a_3 & 0 & a_1 & a_0 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ p_2 \\ p_3 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Padé Approximation: Abstract Example

2 of 2

We get a linear system for p_1, p_2, \dots, p_N and q_1, q_2, \dots, q_N :

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

If we want $n = 3$, $m = 2$:

$$\begin{bmatrix} a_0 & & -1 & & \\ a_1 & a_0 & & -1 & \\ a_2 & a_1 & 0 & & \\ a_3 & a_2 & 0 & 0 & \\ a_4 & a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ p_3 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Padé Approximation: Abstract Example

2 of 2

We get a linear system for p_1, p_2, \dots, p_N and q_1, q_2, \dots, q_N :

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

If we want $n = 3$, $m = 2$:

$$\begin{bmatrix} a_0 & & -1 & & & \\ a_1 & a_0 & & -1 & & \\ a_2 & a_1 & 0 & & -1 & \\ a_3 & a_2 & 0 & 0 & & \\ a_4 & a_3 & 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Padé Approximation: Abstract Example

2 of 2

We get a linear system for p_1, p_2, \dots, p_N and q_1, q_2, \dots, q_N :

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

If we want $n = 3$, $m = 2$:

$$\begin{bmatrix} a_0 & 0 & -1 & & & \\ a_1 & a_0 & 0 & -1 & & \\ a_2 & a_1 & 0 & 0 & -1 & \\ a_3 & a_2 & 0 & 0 & 0 & \\ a_4 & a_3 & 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Padé Approximation: Concrete Example, e^{-x}

1 of 3

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$,
hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

Padé Approximation: Concrete Example, e^{-x}

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$, hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

$$\begin{bmatrix} 1 & 0 & -1 & & & \\ -1 & 1 & 0 & -1 & & \\ 1/2 & -1 & 0 & 0 & -1 & \\ -1/6 & 1/2 & 0 & 0 & 0 & \\ 1/24 & -1/6 & 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1/2 \\ -1/6 \\ 1/24 \\ -1/120 \end{bmatrix},$$

which gives $\{q_1, q_2, p_1, p_2, p_3\} = \{2/5, 1/20, -3/5, 3/20, -1/60\}$

Padé Approximation: Concrete Example, e^{-x}

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$, hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

$$\begin{bmatrix} 1 & 0 & -1 & & & \\ -1 & 1 & 0 & -1 & & \\ 1/2 & -1 & 0 & 0 & -1 & \\ -1/6 & 1/2 & 0 & 0 & 0 & \\ 1/24 & -1/6 & 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1/2 \\ -1/6 \\ 1/24 \\ -1/120 \end{bmatrix},$$

which gives $\{q_1, q_2, p_1, p_2, p_3\} = \{2/5, 1/20, -3/5, 3/20, -1/60\}$, i.e.

$$r_{3,2}(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}.$$

Padé Approximation: Concrete Example, e^{-x}

All the possible Padé approximations of degree 5 are:

$$r_{5,0}(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5$$

$$r_{4,1}(x) = \frac{1 - \frac{4}{5}x + \frac{3}{10}x^2 - \frac{1}{15}x^3 + \frac{1}{120}x^4}{1 + \frac{1}{5}x}$$

$$r_{3,2}(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}$$

$$r_{2,3}(x) = \frac{1 - \frac{2}{5}x + \frac{1}{20}x^2}{1 + \frac{3}{5}x + \frac{3}{20}x^2 + \frac{1}{60}x^3}$$

$$r_{1,4}(x) = \frac{1 - \frac{1}{5}x}{1 + \frac{4}{5}x + \frac{3}{10}x^2 + \frac{1}{15}x^3 + \frac{1}{120}x^4}$$

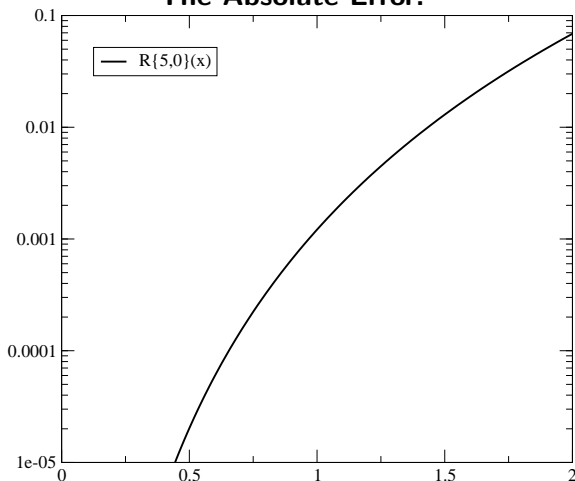
$$r_{0,5}(x) = \frac{1}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5}$$

Note: $r_{5,0}(x)$ is the Taylor polynomial of degree 5.

Padé Approximation: Concrete Example, e^{-x}

3 of 3

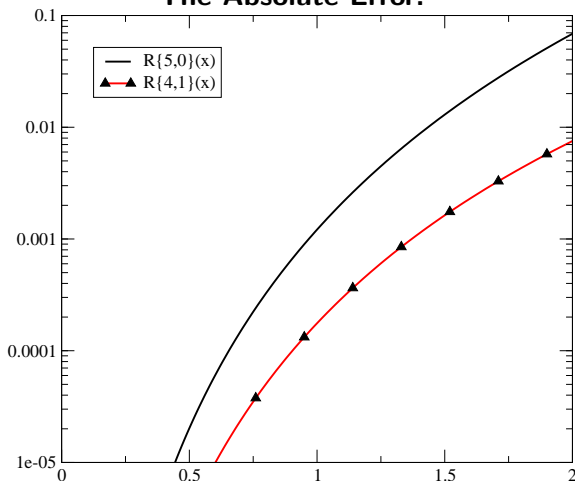
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

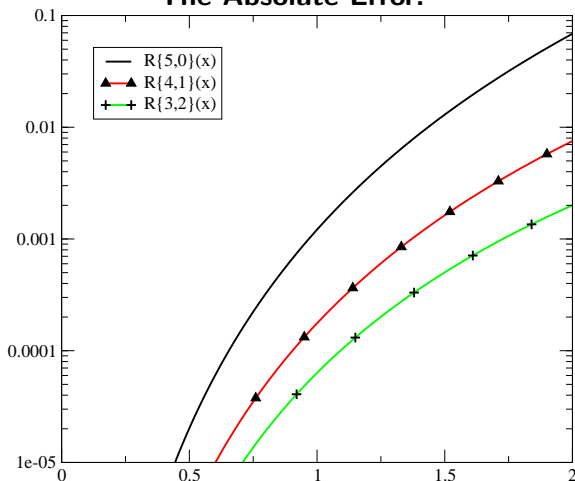
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

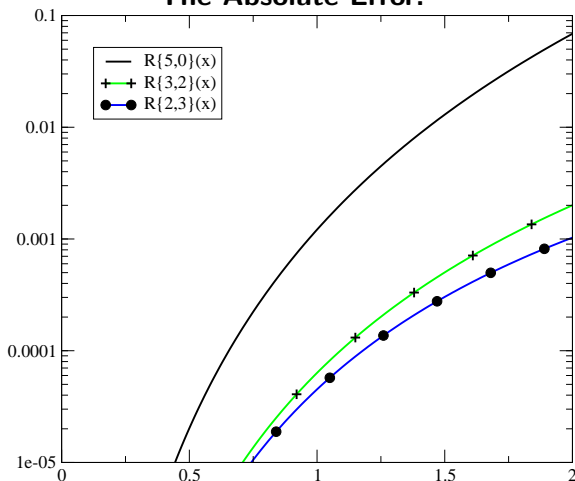
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

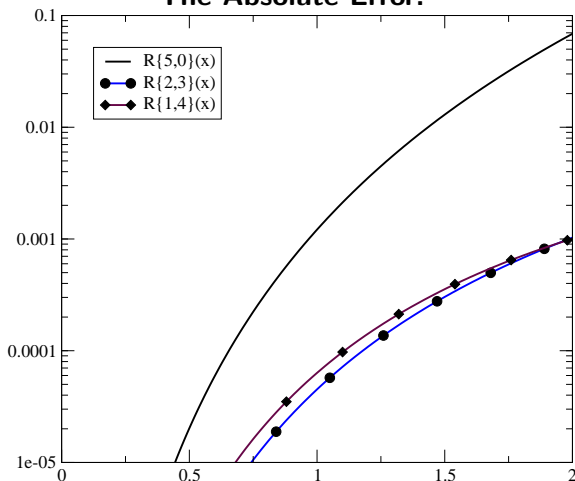
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

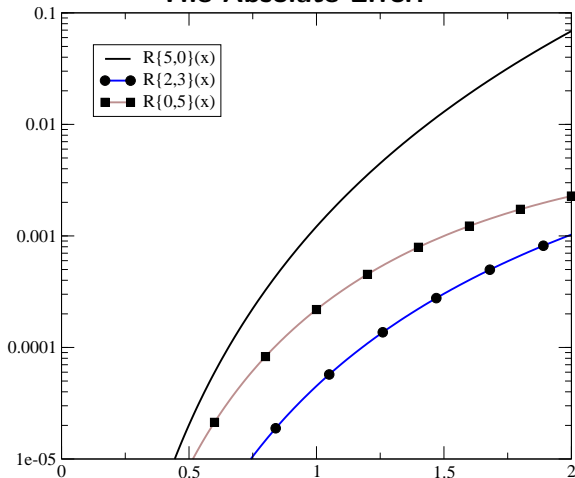
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

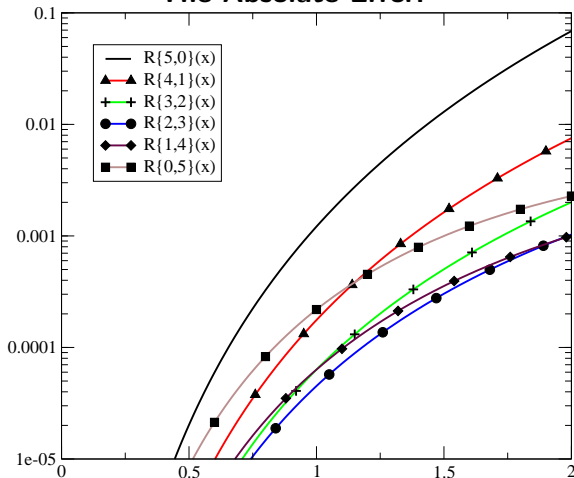
The Absolute Error.



Padé Approximation: Concrete Example, e^{-x}

3 of 3

The Absolute Error.



Padé Approximation: Matlab Code.

The algorithm in the book looks frightening! If we think in term of the matrix problem defined earlier, it is easier to figure out what is going on:

```
% The Taylor Coefficients,  $a_0, a_1, a_2, a_3, a_4, a_5$ 
a = [1 -1 1/2 -1/6 1/24 -1/120]';
N = length(a); A = zeros(N-1,N-1);
% m is the degree of q(x), and n the degree of p(x)
m = 3; n = N-1-m;
% Set up the columns which multiply  $q_1$  through  $q_m$ 
for i=1:m
    A(i:(N-1),i) = a(1:(N-i));
end
% Set up the columns that multiply  $p_1$  through  $p_n$ 
A(1:n,m+(1:n)) = -eye(n)
% Set up the right-hand-side
b = - a(2:N);
% Solve
c = A\b;
Q = [1 ; c(1:m)]; % Select  $q_0$  through  $q_m$ 
P = [a_0 ; c((m+1):(m+n))]; % Select  $p_0$  through  $p_n$ 
```

Optimal Padé Approximation?

	One Point	Optimal Points
Polynomials	Taylor	Chebyshev
Rational Functions	Padé	???

From the example e^{-x} we can see that Padé approximations suffer from the **same problem** as Taylor polynomials – they are very accurate near **one point**, but away from that point the approximation degrades.

“Chebyshev-placement” of interpolating points for polynomials gave us an optimal (uniform) error bound over the interval.

Can we do something similar for rational approximations???

Chebyshev Basis for the Padé Approximation!

We use the **same** idea — instead of expanding in terms of the basis functions x^k , we will use the **Chebyshev polynomials**, $T_k(x)$, as our basis, *i.e.*

$$r_{n,m}(x) = \frac{\sum_{k=0}^n p_k T_k(x)}{\sum_{k=0}^m q_k T_k(x)}$$

where $N = n + m$, and $q_0 = 1$.

Chebyshev Basis for the Padé Approximation!

We use the **same** idea — instead of expanding in terms of the basis functions x^k , we will use the **Chebyshev polynomials**, $T_k(x)$, as our basis, *i.e.*

$$r_{n,m}(x) = \frac{\sum_{k=0}^n p_k T_k(x)}{\sum_{k=0}^m q_k T_k(x)}$$

where $N = n + m$, and $q_0 = 1$.

We also need to expand $f(x)$ in a series of Chebyshev polynomials:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

so that

$$f(x) - r_{n,m}(x) = \frac{\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^m q_k T_k(x) - \sum_{k=0}^n p_k T_k(x)}{\sum_{k=0}^m q_k T_k(x)}.$$

The Resulting Equations

Again, the coefficients p_0, p_1, \dots, p_n and q_1, q_2, \dots, q_m are chosen so that the numerator has zero coefficients for $T_k(x)$, $k = 0, 1, \dots, N$, i.e.

$$\sum_{k=0}^{\infty} a_k T_k(x) \sum_{k=0}^m q_k T_k(x) - \sum_{k=0}^n p_k T_k(x) = \sum_{k=N+1}^{\infty} \gamma_k T_k(x).$$

We will need the following relationship:

$$T_i(x) T_j(x) = \frac{1}{2} [T_{i+j}(x) + T_{|i-j|}(x)].$$

Also, we must compute (maybe numerically)

$$a_0 = \frac{1}{\pi} \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx \quad \text{and} \quad a_k = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx, \quad k \geq 1.$$

Example: Revisiting e^{-x} with Chebyshev-Padé Approximation

1/5

The 8th order Chebyshev-expansion (ALL PRAISE MAPLE) for e^{-x} is

$$\begin{aligned} P_8^{\text{CT}}(x) = & 1.266065878 T_0(x) - 1.130318208 T_1(x) + 0.2714953396 T_2(x) \\ & - 0.04433684985 T_3(x) + 0.005474240442 T_4(x) \\ & - 0.0005429263119 T_5(x) + 0.00004497732296 T_6(x) \\ & - 0.000003198436462 T_7(x) + 0.0000001992124807 T_8(x) \end{aligned}$$

and using the same strategy — building a matrix and right-hand-side utilizing the coefficients in this expansion, we can solve for the Chebyshev-Padé polynomials of degree $(n + 2m) \leq 8$:

Next slide shows the matrix set-up for the $r_{3,2}^{\text{CP}}(x)$ approximation.

Note: Due to the “folding”, $T_i(x)T_j(x) = \frac{1}{2} [T_{i+j}(x) + T_{|i-j|}(x)]$, we need $n + 2m$ Chebyshev-expansion coefficients. (Burden-Faires do not mention this, but it is “obvious” from algorithm 8.2; Example 2 (p. 519) is broken, – it needs $\tilde{P}_7(x)$.)

Example: Revisiting e^{-x} with Chebyshev-Padé Approximation

2/5

$$T_0(x) : \frac{1}{2} \left[\begin{array}{cccc} a_1 q_1 & + & a_2 q_2 & - 2p_0 = 2a_0 \end{array} \right]$$

$$T_1(x) : \frac{1}{2} \left[\begin{array}{cccc} (2a_0 + a_2)q_1 & + & (a_1 + a_3)q_2 & - 2p_1 = 2a_1 \end{array} \right]$$

$$T_2(x) : \frac{1}{2} \left[\begin{array}{cccc} (a_1 + a_3)q_1 & + & (2a_0 + a_4)q_2 & - 2p_2 = 2a_2 \end{array} \right]$$

$$T_3(x) : \frac{1}{2} \left[\begin{array}{cccc} (a_2 + a_4)q_1 & + & (a_1 + a_5)q_2 & - 2p_3 = 2a_3 \end{array} \right]$$

$$T_4(x) : \frac{1}{2} \left[\begin{array}{cccc} (a_3 + a_5)q_1 & + & (a_2 + a_6)q_2 & - 0 = 2a_4 \end{array} \right]$$

$$T_5(x) : \frac{1}{2} \left[\begin{array}{cccc} (a_4 + a_6)q_1 & + & (a_3 + a_7)q_2 & - 0 = 2a_5 \end{array} \right]$$

Example: Revisiting e^{-x} with Chebyshev-Padé Approximation

$$R_{4,1}^{\text{CP}}(x) =$$

$$\frac{1.155054 T_0(x) - 0.8549674 T_1(x) + 0.1561297 T_2(x) - 0.01713502 T_3(x) + 0.001066492 T_4(x)}{T_0(x) + 0.1964246628 T_1(x)}$$

$$R_{3,2}^{\text{CP}}(x) =$$

$$\frac{1.050531166 T_0(x) - 0.6016362122 T_1(x) + 0.07417897149 T_2(x) - 0.004109558353 T_3(x)}{T_0(x) + 0.3870509565 T_1(x) + 0.02365167312 T_2(x)}$$

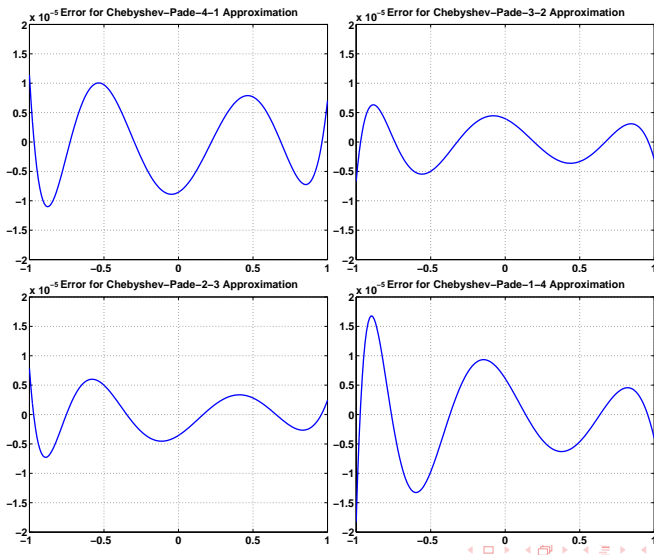
$$R_{2,3}^{\text{CP}}(x) =$$

$$\frac{0.9541897238 T_0(x) - 0.3737556255 T_1(x) + 0.02331049609 T_2(x)}{T_0(x) + 0.5682932066 T_1(x) + 0.06911746318 T_2(x) + 0.003726440404 T_3(x)}$$

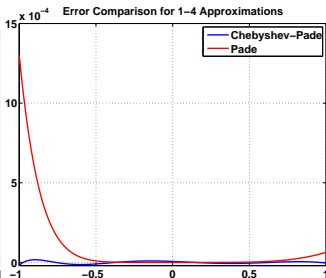
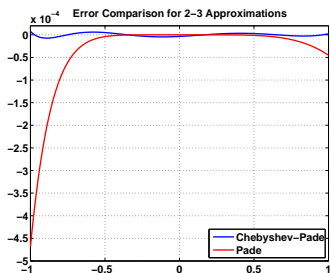
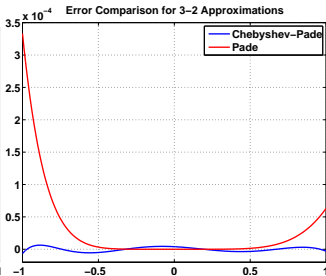
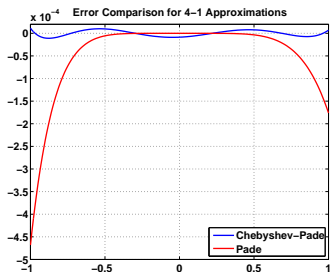
$$R_{1,4}^{\text{CP}}(x) =$$

$$\frac{0.8671327116 T_0(x) - 0.1731320271 T_1(x)}{T_0(x) + 0.73743710 T_1(x) + 0.13373746 T_2(x) + 0.014470654 T_3(x) + 0.00086486509 T_4(x)}$$

Example: Revisiting e^{-x} with Chebyshev-Padé Approximation



Example: Revisiting e^{-x} with Chebyshev-Padé Approximation



The Bad News — It's Not Optimal!

The Chebyshev basis does not give an optimal (in the min-max sense) rational approximation. However, the result can be used as a starting point for **the second Remez algorithm**. It is an iterative scheme which converges to the best approximation.

A discussion of how and why (and why not) you may want to use the second Remez' algorithm can be found in **Numerical Recipes in C: The Art of Scientific Computing** (Section 5.13). [You can read it for free on the web^(*) — just Google for it!]

(*) The old 2nd Edition is Free, the new 3rd edition is for sale...