Math 311

Numerical Methods

1.1: Review of Calculus

Important Theorems for Numerical Methods

S. K. Hyde

Burden and Faires, any ed.

Winter 2025

1 Topics

1.1 Why Review Calculus?

- What do we need Calculus concepts for?
- They help us to develop ways to manipulate and solve problems.
- Either they are unsolvable analytically or we want an easier way (e.g. we're lazy).
- The calculus will help us to make sure that the algorithms make sense.

1.2 Important Theorems and Ideas in Calculus:

- Limits, Continuity, Differentiability, Convergence
- Rolle's Theorem
- The Mean Value Theorm (MVT) (or Weighted MVT for Integrals).
- Extreme Value Theorem (EVT)
- Intermediate Value Theorem (IVT)
- Taylor's Polynomials

Definition of Limit

Definition. *[1.1] Let $f(x)$ be a function defined on a set X of real numbers. f is said to have the* ***limit*** *L at x_0, written $\lim_{x\to x_0} f(x) = L$ if:*

$$\forall\varepsilon > 0, \exists\,\delta(\varepsilon) \text{ such that } |f(x) - L| < \varepsilon \text{ whenever } x \in X \text{ and } 0 < |x - x_0| < \delta(\varepsilon)$$

Definition of Continuous

Definition. *[1.2] Let f be a function defined on a set X and $x_0 \in X$.*

$$\text{If } \lim_{x\to x_0} f(x) = f(x_0), \text{ then } f \text{ is } \textbf{continuous} \text{ at } x_0.$$

Notation of Sets of functions

Definition. *The "C" indicates continuous function, and the power indicates derivatives. So*

- *$C(X)$ is the set of all continuous functions on X.*
- *$C^n(X)$ is the set of all functions having n continuous derivatives on X.*
- *$C^\infty(X)$ is the set of all functions having contin. derivatives of all orders on X.*

Definition of Limit of a sequence

Definition. *[1.3] Let* $\{x_n\}$ *be an infinite sequence of numbers. The sequence is said to converge to a* ***limit x****, if:*

$$\forall \varepsilon > 0, \exists \text{ a positive integer } N(\varepsilon) \text{ such that for all } n > N(\varepsilon) \text{ implies } |x_n - x| < \varepsilon$$

and we write "$\lim_{n \to \infty} x_n = x$*" or "*$x_n \to x$ *as* $n \to \infty$*."*

Definition of Continuous of a sequence

Definition. *[1.4] Let* f *be a function defined on a set* X *of real numbers and let* $x_0 \in X$*. The following are equivilent:*

- f *is* ***continuous*** *at* x_0.
- *If* $\{x_n\}$ *is any infinite sequence converging to* x_0 *(written* $\lim_{n \to \infty} x_n = x_0$*), then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f(x_0)$$

Definitions of Differentiable

Definition. *[1.5] Suppose f is defined on (a, b) and $x_0 \in (a, b)$. Then the function f is **differentiable** at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

When it exists, we define the limit to be the derivative of f at x_0:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

$f'(x_0)$ is the slope of the tangent line to the graph of $f(x)$ at x_0.

Differentiability implies Continuous

Theorem. *[1.6] If f is differentiable at x_0, then f is continuous at x_0.*

Rolle's Theorem

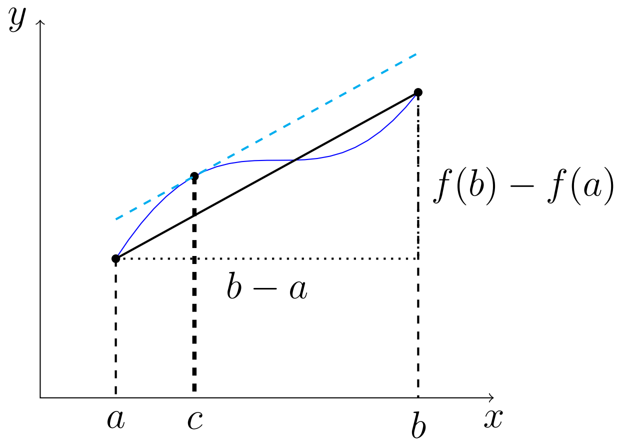
Theorem. *[1.7] Suppose $f \in C[a, b]$ and f' exists on (a, b).*

If $f(a) = f(b) = 0$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem

Theorem. *[1.8] Suppose* $f \in C[a,b]$ *and* f' *exists on* (a,b). *It follows that* $\exists\, c \in (a,b)$ *such that*

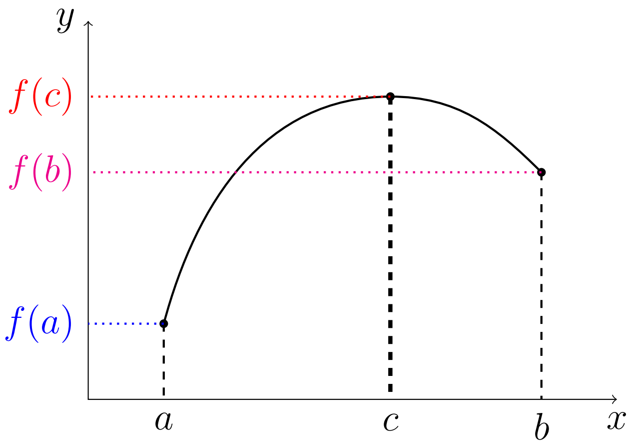
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Extreme Value Theorem

Theorem. *[1.9] Suppose $f \in C[a,b]$. Suppose the mimimizer c_1 and maximizer c_2 over (a,b) exist, which means $f(c_1) \leqslant f(x) \leqslant f(c_2)$ $(\forall x \in [a,b])$.*

If f' exists on (a,b), then c_1 and c_2 occur where $f' = 0$ or at the endpoints (a or b).



The Riemann integral

Definition. *[1.10] The Riemann integral of the function f on the interval $[a, b]$ is the following limit, provided it exists:*

$$\int_a^b f(x)\ dx = \lim_{\max \Delta \boldsymbol{x_i} \to \mathbf{0}} \sum_{i=1}^{n} f(\boldsymbol{z_i}) \boldsymbol{\Delta x_i},$$

where $a = x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = b$, $\boldsymbol{z_i} \in [\boldsymbol{x_{i-1}}, \boldsymbol{x_i}]$, and $\boldsymbol{\Delta x_i = x_i - x_{i-1}}$. Note this says that no matter the spacing, the limit is the same! So let's make it easier to analyze! Let's do even spacing. If the spacing is even, then

$$\boldsymbol{x_i = a + i\Delta x}, \text{ where } \boldsymbol{\Delta x = \frac{b-a}{n}} \text{ and}$$

$$\int_a^b f(x)\ dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(\boldsymbol{z_i}) \boldsymbol{\Delta x}$$

Weighted MVT for Integrals

Theorem. *[1.11] If $f \in C[a,b]$, g is integrable on $[a,b]$, and g does not change sign on $[a,b]$, then there exists a number $c \in (a,b)$ with*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

When $g(x) = 1$, then this gives the average value over $[a,b]$:

$$\int_a^b f(x)dx = f(c)\int_a^b dx = f(c)(b-a).$$

It follows that the average value is

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Generalized Rolle's Theorem

Theorem. *[1.12] Let $f \in C[a,b]$ and $f \in C^n(a,b)$. If f vanishes at the $n+1$ distinct pts $x_0, \cdots, x_n$ in $[a,b]$, then there exists $c \in (a,b)$ such that $f^{(n)}(c) = 0$.*

Intermediate Value Theorem

Theorem. *[1.13] If $f \in C[a,b]$ and k is any number in between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ such that $f(c) = k$.*

In other words, if $f(a) < f(b)$, then

$$f(a) < \quad k \quad < f(b), \text{ then } \exists\, c \in [a,b] \text{ such that}$$
$$f(a) < f(c) < f(b).$$

Example: Show $f(x) = x^5 - 2x^3 + 3x^2 - 1 = 0$ has a root on $[0,1]$.

Note that f is continous and $f(0) = -1$ and $f(1) = 1 - 2 + 3 - 1 = 1$.

$$\text{Since } -1 = f(0) < \quad 0 \quad < f(1) = 1, \text{ then } \exists\, c \in [0,1] \text{ such that}$$
$$-1 = f(0) < f(c) < f(1) = 1.$$

Thus, f has a root on $[0,1]$ ($f(c) = 0$).

So how about the Generalized Mean Value Theorem? That's also known as Taylor's Theorem!!!

Taylor's Theorem (Thm 1.14)

Theorem. *Suppose* $f \in C^n[a,b]$, $f^{(n+1)}$ *exists on* $[a,b]$, *and* $x_0 \in [a,b]$. *For every* $x \in [a,b]$, *there exists* $\xi(x)$ *between* x_0 *and* x *(meaning either* $\xi(x) \in (x_0, x)$ *or* $\xi(x) \in (x, x_0)$*) with*

$$\boldsymbol{f(x) = P_n(x) + R_n(x)}, \text{ where}$$

$$\boldsymbol{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k} \text{ and } \boldsymbol{R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}}.$$

- This theorem is EXTREMELY IMPORTANT for numerical analysis and is used MANY times in the development of procedures. You must have a good basic understanding of the idea.
- When $n = 1$, this simplifies to the Mean Value Theorem, so consider this an extension of that theorem. It truly is the "Generalized Mean Value Theorem" (similar to the Generalized Rolle's Theorem).
- Note that the remainder term ($R_n(x)$) is the first neglected term in the infinite series, but $f^{(n+1)}$ is evaluated at the sweet spot $\xi(x)$ which maintains equality!

- When $n \to \infty$, $P_n(x)$ converges to the Taylor Series for $f(x)$. However, this requires $f \in C^\infty[a,b]$.
- Let $x_0 = 0$. We refer to these as "Maclaurin" series. (Although "Taylor's" is used just as much even when $x_0 = 0$)
- Two forms: (Call them "regular form" and "h-form")

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots \tag{1}$$

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots$$

- To switch between the two, just remember

$$x = x_0 + h$$

- It's more common for the subscript of x_0 to be dropped in the "h-form":

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \tag{2}$$

Example:

Suppose $f(x) = e^x$, (so $f^{(k)}(x) = e^x$). Let $x_0 = 0$.
Taylor's theorem with $n = 2$ is thus:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(\xi)(x - x_0)^2$$

$$f(x) = f(0) + f'(0)(x - 0) + f''(\xi)\frac{(x-0)^2}{2!}$$

$$f(x) = 1 + x + f''(\xi)\frac{x^2}{2!}$$

Estimate the error in an interval

Suppose we want to estimate the error over $[-1, 1]$. By Def, the error is

$$f(x) - P_1(x) = R_1(x) = \frac{f''(\xi)x^2}{2} = \frac{e^{\xi}x^2}{2}$$

Next, we use the Extreme Value Theorem to find an upper bound. $f(x) = e^x$ has no turning points in $[-1, 1]$, and since $-1 < \xi(x) < 1$, then $e^{-1} < e^{\xi} < e^1$. So an upper bound on the error is $ex^2/2$. Thus, for $x \in [-1, 1]$,

$$e^x \approx 1 + x + kx^2, \text{ where } k = e/2, \text{ and } x \text{ is near zero.}$$

"Big-O" Notation

Definition. $a_n = \mathcal{O}(b_n)$ *(Read as: a_n is big O of b_n) if the ratio $|a_n/b_n|$ is* ***bounded*** *for large n; in detail, if there exists a number K and an integer $N(K)$ such that for all $n > N(K)$, it follows that*

$$|a_n| < K|b_n|.$$

An equivalent definition is that $f(x) = \mathcal{O}(g(x))$ as $x \to 0$ means

$$|f(x)| \leqslant c|g(x)|,$$

whenever x is sufficiently small. Note that in either case, we talk about terms where $n \to \infty$ or $x \to 0$. Otherwise it doesn't make sense.

For example, the following are equivalent

$$e^x = 1 + x + cx^2 \qquad\qquad e^x = 1 + x + \mathcal{O}\left(x^2\right)$$

This allows us to work with series and only use a finite number of terms to do our work. We keep only the most significant parts and gather all the other parts into the Big-O term. We also have another notation, called "Little-o" notation, which is defined in a similar manner:

"Little-O" Notation

Definition. $a_n = \mathcal{o}(b_n)$ *(Read as: a_n is little o of b_n) if the ratio $|a_n/b_n|$ converges to 0; in detail, if, for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that if n exceeds $N(\varepsilon)$, then*

$$|a_n| < \varepsilon |b_n|.$$

An equivalent definition is that $f(x) = \mathcal{o}(g(x))$ as $x \to 0$ means

$$\frac{|f(x)|}{|g(x)|} \to 0$$

as $x \to 0$. Note that in either case, we talk about terms where $n \to \infty$ or $x \to 0$.

It is difficult to tell the difference between writing $\mathcal{o}(x)$ and writing $\mathcal{O}(x)$ on paper, so I will try and emphasize the size of the o when writing.

Note that "Big-O" contains MORE INFORMATION about the relation between a_n and b_n, so is preferred where possible.

Additional Assignments for Section 1.1-1.3

- There is an additional homework assignment on Canvas entitled "Taylor Series / Big-O Assignment" with a paper for you to read called "Taylor Series, Taylor Polynomials, and Big-O Notation".
- Read through the whole document and answer the questions in the Exercises. Scan your solutions to the exercises and submit on Canvas.
- There are additional examples in the paper to help you understand Big-O, little-o notation. Be prepared to ask questions in class.
- If you work through that document, and ask questions in class for it, you will have a good understanding of the notation.