Math 311

Numerical Methods

1.3: Algorithms and Convergence

Taylor Series, Big-O-Notation

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Introduction

Algorithms

Definition.

- *An algorithm is a procedure that describes, in an unambiguous manner, a finite sequence of steps to be performedin a specific order.*
- *The object is to implement a numerical procedure to solve a problem or approximate a solution to the problem.*

Pseudocode

Definition.

- *Pseudocode describes the steps of an algorithm.*
- *Uses **basic programming control flow** mixed with english*
- *Note that the code needs to be finite, otherwise the program will never finish.*
- *We are not that good yet to complete a infinite task in a finite amount of time.*

Basic Program Control Flow

Definition. *You can split decisions and steps using the following basic ideas:*

- *Decision making*
- *Definite Looping*
- *Indefinite Looping*

0.1 Decision Making

Part of the flow of an algorithm (or program) is making decisions and taking action. The main construct for this is the "If-Then-Else" construct.

- If [a condition is true]; then [take an action]; endif
- If [a condition is true]; then [take an action]; else (take a different action when false); endif

0.2 Definite Looping

Definite Looping is the process of repeating an action a specific number of times.

- Construct is "For [var] = 1, 2, 3, ..., n". This performs the action the specified number of times (in this case n times).

0.3 Indefinite Looping

Indefinite Looping is the process of repeating an action an indefinite number of times until a condition is met. This is usually combined with a condition to check that will stop the loop if met (or continue a loop if the condition is met (either way).

- Construct is "While [a condition is true] do [these steps]"
- This will continue a loop as long as the beginning condition is met.
- Another construct is "Repeat [these steps] until [a condition is true]". This is similar to the above. Only difference is that a "Repeat" statement will always be executed once (and the check is at the end instead of the beginning).

- In Numerical Analysis, power series are used throughout the development of many numerical procedures.
- Examples of power series include Taylor Series and Maclaurin Series.
- Maclaurin Series are defined as Taylor Series centered about 0.
- Loosely, we will call them all "Taylor Series"

1 Taylor Series

Definition. ***Sets of Functions.***

- $C(X)$ *- set of all continuous functions on the set* X*.*
- $C^1(X)$ *- set of all functions having* ***a continuous derivative*** *on the set* X*.*
- $C^n(X)$ *- set of all functions having* n ***continuous derivatives*** *on the set* X*.*
- $C^\infty(X)$ *- set of all functions having continuous derivatives of all orders on* X*.*

Taylor's Theorem (Thm 1.14)

Theorem. *Suppose* $f \in C^n[a,b]$, $f^{(n+1)}$ *exists on* $[a,b]$, *and* $x_0 \in [a,b]$. *For every* $x \in [a,b]$ *there exists* $\xi(x)$ *between* x_0 *and* x *with*

$$f(x) = P_n(x) + R_n(x), \text{ where}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \qquad R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \qquad (1)$$

- Note that $P_n(x)$ is called the n**th Taylor polynomial** for f about x_0
- $R_n(x)$ is called the **remainder term** (or **trunctation error**) associated with $P_n(x)$.
- The limit (as $n \to \infty$) of $P_n(x)$ is called the **Taylor series** for f about x_0.
- If $x_0 = 0$, then the Taylor polynomial is called a **Maclaurin polynomial** and the Taylor Series is called a Maclaurin Series. (In practice, both are loosely called the Taylor Series or Power Series)

Not all functions have a power series representation (or Taylor Series representation). For example, the function

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- You can check and you will find out that this function is in $C^\infty(\mathbb{R})$.
- However, every derivative of $f(x)$ is zero at the origin!
- This means that $f^{(n)}(0) = 0$ for all $n \geq 0$.
- Thus, if we tried to form the Taylor Series of it at 0, we would get $f(x) = 0$, which it is not!

When a function at a point can be expressed as a power series, then we say the function is **analytic**

2 Big-O Notation

Big-O Notation

Definition.

$$a_n = \mathcal{O}(b_n) \text{ (Read: } a_n \text{ is big O of } b_n\text{) if the ratio } \left|\frac{a_n}{b_n}\right| \text{ is bounded for large } n. \tag{2}$$

Little-o Notation

Definition.

$$a_n = o(b_n) \text{ (Read: } a_n \text{ is little o of } b_n\text{) if the ratio } \left|\frac{a_n}{b_n}\right| \text{ converges to zero.} \tag{3}$$

- The idea behind these definitions is to compare the approximate size or order of magnitude of $\{a_n\}$ to that of $\{b_n\}$.
- In most applications, $\{a_n\}$ is the sequence of interest and $\{b_n\}$ is a comparison sequence.

Definition. ***Big-O Notation*** *for functions*

$f(x) = \mathcal{O}(g(x))$ *as* $x \to L$	*if for any sequence* $\{x_n\}$ *such that* $x_n \to L$*, we have* $f(x_n) = \mathcal{O}(g(x_n))$ *in the sense of the definition of Big-O above.*

Definition. ***Little-o Notation*** *for functions*

$f(x) = \mathcal{O}(g(x))$ *as* $x \to L$	*if for any sequence* $\{x_n\}$ *such that* $x_n \to L$*, we have* $f(x_n) = o(g(x_n))$ *in the sense of the definition of Little-O above.*

- Think of Big-O notation as meaning that $f(x)$ and $g(x)$ are the "same size".
- In practice, we can use the following proposition (which is equivilent for applications we are interested in).

Corollary. *If*

$$\lim_{x \to L} \left| \frac{f(x)}{g(x)} \right| = k < +\infty, \text{ then } f(x) = \mathcal{O}(g(x)) \text{ (read as "}f\text{ is big-O of }g\text{").} \tag{4}$$

- *Note that when $k = 0$, then $f(x) = o(g(x))$. (little-o)*
- *However, we are most interested when $k \neq 0$, because that conveys more information about the relation between $f(x)$ and $g(x)$.*

- When it is clear, the condition "as $x \to L$" is not always explicity stated.
- The value of L can be any number (or also $\pm\infty$).

2.1 Combinations of Big-O terms

- Sometimes we want to combine terms that are either Big-O or little-o together.
- Here are some rules that can easily be proven by using the definitions above.

Combinations of Big-O (or little-o) terms.

Definition.

- $\mathcal{O}(ca_n) = \mathcal{O}(a_n)\,, c \in \mathbb{R}, c \neq 0$
- $a_n = \mathcal{O}(a_n)$
- $a_n = \mathcal{o}(1) \implies \lim_{n\to\infty} a_n = 0$
- $a_n = \mathcal{O}(1) \implies |a_n| < K.$
- $a_n\mathcal{o}(1) = \mathcal{o}(a_n)$
- $a_n\mathcal{O}(1) = \mathcal{O}(a_n)$
- $\mathcal{O}(a_n)\,\mathcal{O}(b_n) = \mathcal{O}(a_n b_n)$
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- $\mathcal{O}(\mathcal{o}(a_n)) = \mathcal{o}(a_n)$
- $\mathcal{o}(\mathcal{o}(a_n)) = \mathcal{o}(a_n)$
- $\mathcal{o}(\mathcal{O}(a_n)) = \mathcal{o}(a_n)$
- $\mathcal{O}(a_n) + \mathcal{O}(b_n) = \mathcal{O}(\max(|a_n|, |b_n|))$
- $\mathcal{o}(a_n) + \mathcal{o}(b_n) = \mathcal{o}(\max(|a_n|, |b_n|))$
- $\mathcal{O}(a_n) + \mathcal{o}(b_n) = \begin{cases} \mathcal{O}(a_n)\,, \textit{if } b_n = \mathcal{o}(a_n) \;\; \textit{or } b_n = \mathcal{O}(a_n) \\ \mathcal{O}(b_n)\,, \textit{if } a_n = \mathcal{O}(b_n) \\ \mathcal{o}(b_n)\,, \textit{if } a_n = \mathcal{o}(b_n) \end{cases}$

2.2 Big-O Examples

What above the statement $5 = \mathcal{O}(1)$? Is it true that 5 is big-O of 1? By the corollary above, yes it is since $\lim_{x \to L} \left| \frac{5}{1} \right| = 5 < \infty$. Note that in this case, L doesn't matter as it works for any value of L.

- $3x^2 = \mathcal{O}\big(x^2\big)? \implies \lim_{x \to L} \left| \frac{3x^2}{x^2} \right| = 3 < \infty$, so, yes, $3x^2$ is big-O of x^2 (as $x \to 0$).
- $\sin x = \mathcal{O}(1)? \implies \lim_{x \to L} \left| \frac{3\sin(x)}{1} \right| = |\sin L| < \infty$.
  - Note that for all L, $|\sin(L)| = K < \infty$, so $\sin x = \mathcal{O}(1)$.
  - Note also that if $L = 0$, then it follows that $\sin x = o(1)$.
- $\sin x = \mathcal{O}(x)? \implies \lim_{x \to 0} \left| \frac{\sin x}{x} \right| \overset{\text{L.R.}}{=} \lim_{x \to 0} \left| \frac{\cos x}{1} \right| = 1 < \infty$, so, yes, $\sin x$ is big-O of x (as $x \to 0$).
- $\sin x \neq \mathcal{O}\big(x^2\big)? \implies \lim_{x \to 0} \left| \frac{\sin x}{x^2} \right| \overset{\text{L.R.}}{=} \lim_{x \to 0} \left| \frac{\cos x}{2x} \right| = \infty$, so, NO, $\sin x$ is not big-O of x^2 (as $x \to 0$).

- $\sin x - x = o\left(x^2\right)? \Longrightarrow \lim_{x\to 0}\left|\frac{\sin x - x}{x^2}\right| \overset{\text{L.R.}}{=} \lim_{x\to 0}\left|\frac{\cos x - 1}{2x}\right| \overset{\text{L.R.}}{=} \lim_{x\to 0}\left|\frac{\sin x}{2}\right| = 0 < \infty,$
  so, yes, $\sin x$ is little-o of x^2. Note that it is ALSO $\mathcal{O}\left(x^2\right)$ at the same time.
- Simplify $3x^2 + \mathcal{O}\left(x^2\right) + \mathcal{O}\left(x^3\right)$.
  - We already know that $3x^2 + \mathcal{O}\left(x^2\right) = \mathcal{O}\left(x^2\right)$.
  - Next, $\mathcal{O}\left(x^2\right) + \mathcal{O}\left(x^3\right) = \mathcal{O}\left(x^2 + |x^3|\right) = \mathcal{O}\left(x^2 \underbrace{(1 + |x|)}_{c}\right) = \mathcal{O}\left(cx^2\right) = \mathcal{O}\left(x^2\right)$.

2.3 Examples

- Big-O is used in scenarios where $n \to \infty$ or $x \to 0$.
- The rules in the definition above help with Big-O manipulation.
- Think of Big-O as combining all terms of a certain size or smaller.
- Note, that, as $\underline{\underline{x \to 0}}$, x^7 is LARGER than x^9, x^{11}, etc., whereas, when $n \to \infty$, then n^7 is bigger than n^2.
- Computer scientists are interested in analyzing the complexity of algorithms.
- For example, Bubble Sort is $\mathcal{O}\left(n^2\right)$ and Merge Sort is $\mathcal{O}(n \log n)$. In this case, we can see that Merge Sort is a quicker algorithm.

Here are some examples using some of the properties above.

$$x + \mathcal{O}(x) = \mathcal{O}(x) \qquad x\mathcal{O}(x) = \mathcal{O}\left(x^2\right)$$

$$\mathcal{O}\left(x^2\right) + \mathcal{O}\left(x^4\right) = \mathcal{O}\left(x^2\right) \qquad \mathcal{O}\left(x^2\right)\mathcal{O}\left(x^3\right) = \mathcal{O}\left(x^5\right)$$

$$\mathcal{O}\left(17x^3\right) = \mathcal{O}\left(x^3\right) \qquad \mathcal{O}\left(n^{-1}\right) + \mathcal{O}\left(n^{-3/2}\right) = \mathcal{O}\left(n^{-1}\right)$$

$$\frac{1}{n}\mathcal{O}(1) = \mathcal{O}\left(n^{-1}\right) \qquad 1{,}000{,}000 \cdot \mathcal{O}(n \ln n) = \mathcal{O}(n \ln n)$$

One very important mistake not to make when combining Big-O terms:

$$\mathcal{O}\left(x^3\right) - \mathcal{O}\left(x^3\right) \neq 0 \text{ (Not necessarily - it might be zero)}$$

- The reason for that is that we don't know the constant factor for the two terms – they may or may not cancel each other out.
- This is a common mistake when first using the notation.
- A good way to not make the mistake is to assume that every Big-O term is positive and never negative.
- So, for example,

$$-\frac{1}{2}\mathcal{O}\left(x^2\right) = \mathcal{O}\left(x^2\right) \neq -\mathcal{O}\left(x^2\right).$$

Taylor's Theorem (Using Big-O) (Thm 1.14)

Theorem.
Suppose $f \in C^{n+1}[a,b]$ *and* $x_0 \in [a,b]$. *The Taylor expansion can be written as*

$$f(x) = P_n(x) + \mathcal{O}\big((x-x_0)^{n+1}\big)\,, \quad \textit{where } P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Big-O notation becomes useful when we work with series For example,

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \mathcal{O}\big(x^9\big) \text{ or} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}\big(x^7\big) \text{ or} \\ &= x - \frac{x^3}{3!} + \mathcal{O}\big(x^5\big) \text{ or} \\ &= x + \mathcal{O}\big(x^3\big) \text{ or} \\ &= \mathcal{O}(x) \text{ or } \mathcal{O}(1)\end{aligned}$$

- What Big-O allows you to do is keep the parts that are important (large) and toss out tiny things not needed for a calculation.

2.4 Example 2

- For example, suppose you are interested in the power series for $\sin x \cos x$.
- One way to figure that out is to multiply the two series together:

$$\sin x \cos x = \left[\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\right] \left[\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\right]$$
$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right] \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right]$$

- This is an infinite multiplication! Very complicated!
- However, suppose we are interested only in the accuracy to $\mathcal{O}\left(x^9\right)$.
- Then we can reduce the number of multiplications needed by keeping only the terms that we need.
- You do every possible multiplication of terms, but you drop any that have x^9 or higher powers.

- So the problem changes to a simpler one. Think of all the ways to combine a term on the left with a term on the right.

$$\sin x \cos x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \mathcal{O}\big(x^9\big)\right]\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \mathcal{O}\big(x^8\big)\right]$$

$$\begin{aligned}
= x \quad & -\frac{1}{2!}x^3 \quad + \frac{1}{4!}x^5 \quad - \frac{1}{6!}x^7 \quad + \mathcal{O}\big(x^9\big) \\
& -\frac{1}{3!}x^3 \quad + \frac{1}{3!2!}x^5 \quad - \frac{1}{3!4!}x^7 \quad + \mathcal{O}\big(x^9\big) \\
& \qquad\qquad + \frac{1}{5!}x^5 \quad - \frac{1}{5!2!}x^7 \quad + \mathcal{O}\big(x^9\big) \\
& \qquad\qquad\qquad\qquad - \frac{1}{7!}x^7 \quad + \mathcal{O}\big(x^9\big) \\
\hline
= x \quad & -\frac{2}{3}x^3 \quad + \frac{2}{15}x^5 \quad - \frac{4}{315}x^7 \quad + \mathcal{O}\big(x^9\big)
\end{aligned}$$

$$\sin x \cos x = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \mathcal{O}\big(x^9\big)$$

- Now the problem is not as difficult!!

2.5 Example 3

Without using L'Hôpital's Rule, show

$$\lim_{x\to 0} \frac{2x\arctan x - \log(1+x^2) - x^2}{x^2(1-\cos x)} = -\frac{1}{3}$$

We will take only two terms of the series for $\arctan x$, $\log(1+x^2)$, and $\cos x$. So it follows that they are

$$\begin{aligned} \arctan x &= x - \frac{1}{3}x^3 + \mathcal{O}\left(x^5\right) \\ \log(1+x^2) &= x^2 - \frac{1}{2}x^4 + \mathcal{O}\left(x^6\right) \\ 1-\cos x &= \frac{1}{2}x^2 - \frac{1}{24}x^4 + \mathcal{O}\left(x^6\right) \end{aligned} \tag{5}$$

Using (5), the problem proceeds as follows:

$$\lim_{x\to 0} \frac{2x\arctan x - \log(1+x^2) - x^2}{x^2(1-\cos x)} = \lim_{x\to 0} \frac{2x\left[x - \frac{1}{3}x^3 + \mathcal{O}\left(x^5\right)\right] - \left[x^2 - \frac{1}{2}x^4 + \mathcal{O}\left(x^6\right)\right] - x^2}{x^2\left[\frac{1}{2}x^2 - \frac{1}{24}x^4 + \mathcal{O}(x^6)\right]}$$

Note that in the normal multiplication of the terms above, we have $\mathcal{O}\left(x^6\right) - \mathcal{O}\left(x^6\right)$. Remember, this does not cancel. Think of it like $ax^6 - bx^6 = (a-b)x^6$. Only when

$a = b$ do we have cancellation. So, continuing

$$= \lim_{x\to 0} \frac{2x\left[x - \frac{1}{3}x^3 + \mathcal{O}\left(x^5\right)\right] - \left[x^2 - \frac{1}{2}x^4 + \mathcal{O}\left(x^6\right)\right] - x^2}{x^2\left[\frac{1}{2}x^2 - \frac{1}{24}x^4 + \mathcal{O}(x^6)\right]}$$

$$= \lim_{x\to 0} \frac{\cancel{2x^2} - \frac{1}{3}x^4 + \mathcal{O}\left(x^6\right) - \cancel{x^2} + \frac{1}{2}x^4 + \mathcal{O}\left(x^6\right) - \cancel{x^2}}{x^2\left[\frac{1}{2}x^2 - \frac{1}{24}x^4 + \mathcal{O}(x^6)\right]} \qquad \text{(cancel terms)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{3}x^4 + \mathcal{O}\left(x^6\right) + \frac{1}{2}x^4 + \mathcal{O}\left(x^6\right)}{\frac{1}{2}x^4 - \frac{1}{24}x^6 + \mathcal{O}(x^8)} \qquad \text{(combine terms)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{6}x^4 + \mathcal{O}\left(x^6\right)}{\frac{1}{2}x^4 + \mathcal{O}(x^6)} \qquad \text{(Now multiply top and bottom by } \tfrac{2}{x^4}\text{)}$$

$$= \lim_{x\to 0} \frac{-\frac{2}{6} + \mathcal{O}\left(x^2\right)}{1 + \mathcal{O}(x^2)} \qquad \text{(now evaluate the limit)}$$

$$= -\frac{1}{3}$$

If you try to do this using L'Hôpital's Rule, you'll see that this is much more difficult.

3 Exercises

1. Prove (or disprove) the following statements

   (a) $7 = \mathcal{O}(1)$ as $x \to 0$.

   (b) $x = \mathcal{O}(x)$ as $x \to 0$.

   (c) $x = o(x)$ as $x \to 0$.

   (d) $x = o(1)$ as $x \to 0$.

   (e) $x = \mathcal{O}(e^x)$ as $x \to 0$.

   (f) $\cos x = \mathcal{O}(x)$ as $x \to 0$.

   (g) $\tan x = \mathcal{O}(x)$ as $x \to 0$.

   (h) $n^{-1/2} = \mathcal{O}(n)$ as $n \to \infty$.

   (i) $987x^2 + 7x^5 + x^{11} = \mathcal{O}\left(x^3\right)$ as $x \to 0$

   (j) $x \log x = \mathcal{O}\left(x^2\right)$ as $n \to \infty$.

   (k) $e^x = \mathcal{O}(1)$ as $x \to 0$.

2. Simplify the following expressions (Assume that $x \to 0$.)

   (a) $\mathcal{O}\left(x^2\right) + \mathcal{O}\left(x^4\right) + x + x^6$

   (b) $x\mathcal{O}\left(x^5\right)$

   (c) $987x^7 + \mathcal{O}\left(x^6\right)$

   (d) $\mathcal{O}\left(x^2\right) o\left(x^2\right)$

   (e) $\mathcal{O}\left(x^3\right) o\left(x^2\right)$

   (f) $\mathcal{O}\left((x+1)^3\right)$

   (g) $\mathcal{O}\left((1+n^{-1})^2\right)$

   (h) $\mathcal{O}(1) + o(x)$

   (i) $\mathcal{O}\left(x\mathcal{O}\left(x^5\right)\right)$

   (j) $\cos x - 1 + \frac{1}{2}x^2$ (expand as appropriate)

   (k) $\left(x + 3x^3 + \mathcal{O}\left(x^5\right)\right)^3$ (Note this is easy if you figure the smallest Big-O power first (suppose it is $\mathcal{O}(x^b)$). Then any terms that have that power or larger (x^a, where $a \geqslant b$) will be absorbed into it.)

3. Calculate the following limits using Big-O notation.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + 2x^3}{x^2}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\log(1 + x\arctan x) - e^{x^2} + 1}{\sqrt{1+x^4} - 1}$

   (c) $\displaystyle\lim_{x\to 0} \frac{24 - 24\cos(\sin(x)) - 12x^2 + 5x^4}{\sin(1-\cos(x)) - 1 + \cos x}$

4. Find the first 4 terms for the power series for $e^x \sin x$ (centered at 0).

5. Find the first 4 terms for the power series for $\dfrac{x}{1-x}$ (centered at 0).

6. Find the first two terms for the power series expansion of the sequence

$$x_{k+1} = 1 - \frac{2\sqrt{1-x_k}}{2-x_k}.$$

4 Common Taylor Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{(for all } x\text{)}$$

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad (\text{ for } -1 < x \leqslant 1)$$

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad (\text{ for } -1 \leqslant x < 1)$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} \qquad (\text{ for } -1 < x < 1)$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} x^k \qquad (\text{ for } |x| < 1)$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} (-1)^k x^k \qquad (\text{ for } |x| < 1)$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad (\text{ for } |x|<1)$$

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{4^k(2k-1)} \binom{2k}{k} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$$(\text{ for } |x|<1)$$

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$(\text{for all } x)$$

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$(\text{for all } x)$$

$$\tan x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} x^{2k-1} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

$$(|x| < \tfrac{\pi}{2})$$

$$\csc x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}2(2^{2k-1}-1)B_{2k}}{(2k)!} x^{2k-1} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots$$

$$(|x| < \tfrac{\pi}{2})$$

$$\sec x = \sum_{k=1}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} x^{2k} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots \quad (|x| < \tfrac{\pi}{2})$$

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 + \cdots \quad (|x| < \tfrac{\pi}{2})$$

$$\arcsin x = \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \binom{2k}{k} x^{2k+1} \quad (|x| \leqslant 1)$$

$$\arccos x = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \binom{2k}{k} x^{2k+1} \quad (|x| \leqslant 1)$$

$$\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} x^{2k-1} \quad (|x| \leqslant 1)$$

- B_k is the k^{th} Bernoulli number
- E_k is the k^{th} Euler number.