Math 311

Numerical Methods

1.3b: Algorithms and Convergence

Convergence Example

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Convergence of Sequences: Example

Lemma. *The first order Taylor Polynomial of* $\log(1+t)$ *(centered at 0) is*

$$\log(1+t) = t + \mathcal{O}\left(t^2\right) \tag{1}$$

Proof. Suppose $f(t) = \log(1+t)$. The first derivative of $f(t)$ is $f'(t) = \dfrac{1}{1+t}$. The first order Taylor polynomial centered at 0 is

$$\begin{aligned} f(t) &= f(0) + f'(0)t + \mathcal{O}\left(t^2\right) \\ \log(1+t) &= 0 + (1)t + \mathcal{O}\left(t^2\right) \\ \log(1+t) &= t + \mathcal{O}\left(t^2\right) \end{aligned}$$

□

Next, we want to create an example sequence to analyze from this. Let's define

$$e_n = \left(1 + \frac{1}{n}\right)^n \tag{2}$$

We want to show what this converges to, and the speed at which it does.

Theorem. *The sequence* e_n *converges to* e.

Proof. Using our Lemma, when $t = \frac{1}{n}$, it follows that

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \qquad \text{(Multiply both sides by } n\text{)}$$

$$\boxed{x_n = n\log\left(1+\frac{1}{n}\right) = 1 + \mathcal{O}\left(\frac{1}{n}\right)} \tag{3}$$

This means that

$$\log e_n = n\log\left(1+\frac{1}{n}\right) = 1 + \mathcal{O}\left(\frac{1}{n}\right) \to 1 \qquad (\text{as } n \to \infty).$$

The limit of e_n simplifies as follows:

$$\lim_{n\to\infty} e_n = \lim\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{x_n} = e^{\lim x_n} = e^1 = e.$$

Thus the limit of the sequence e_n is e, as required. □

- How quickly does the sequence converge to e?
- So far, all we know is that it converges at the rate of $\mathcal{O}\left(\frac{1}{n}\right)$.
- We'd like to speed up the convergence of (**??**), so we will develop a new faster converging sequence.

- Consider the sequence

$$w_n = (n + c)x_n,$$

where c is a constant to be determined that will speed up the convergence of the sequence x_n to second order.

- For this, we will need the second order Taylor Polynomial for $\log(1 + t)$.

Lemma. *The second order Taylor Polynomial of* $\log(1 + t)$ *(centered at 0) is*

$$\log(1 + t) = t - \frac{1}{2}t^2 + \mathcal{O}\left(t^3\right)$$

Proof. Continuing from the previous lemma, we have the second derivative of $f(x)$ is $f''(x) = -\frac{1}{(1+t)^2}$. It follows that the second Taylor Polynomial is

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \mathcal{O}\left(t^3\right)$$

$$\boxed{\log(1 + t) = t - \frac{1}{2}t^2 + \mathcal{O}\left(t^3\right)}$$

□

By the above Lemma, it follows that x_n can be written as

$$x_n = \log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

Now, here's the trick to improving our estimates. Find the constant c which improves the Big-O error of x_n (**??**) from $\mathcal{O}\left(\frac{1}{n}\right)$ to $\mathcal{O}\left(\frac{1}{n^2}\right)$. Watch:

$$\begin{aligned}
w_n = (n+c)x_n &= (n+c)\log\left(1 + \frac{1}{n}\right) \\
&= (n+c)\left[\frac{1}{n} - \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] \\
&= n\left[\frac{1}{n} - \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] + c\left[\frac{1}{n} - \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] \\
&= 1 - \frac{1}{2n} + c\frac{1}{n} - c\frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^2}\right) \qquad \text{(now combine terms)}
\end{aligned}$$

$$\boxed{w_n = 1 + \left(c - \frac{1}{2}\right)\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)}$$

By choosing $c = \frac{1}{2}$, we can speed up the convergence of w_n from $\mathcal{O}\left(\frac{1}{n}\right)$ to $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Therefore, $w_n = \left(n + \frac{1}{2}\right) x_n$ converges faster than x_n which means the sequence

$$e_n^{(1)} = e^{w_n} = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$$

converges to e faster than e_n. In particular, they converge at these rates:

$$e_n = \exp\left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\} \text{ and } e_n^{(1)} = \exp\left\{1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}!$$

Let's speed it up one more time. This time, consider the sequence

$$\left(n + c + \frac{d}{n}\right) x_n,$$

where c and d are constants to be determined that will speed up the convergence. Next, we start with a third order Taylor Polynomial for $\log(1 + t)$

Lemma. *The third order Taylor Polynomial of* $\log(1+t)$ *(centered at 0) is*

$$\log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \mathcal{O}\left(t^4\right)$$

Proof. Continuing from the previous lemma, we have the third derivative of $f(x)$ is $f'''(x) = \frac{2}{(1+t)^3}$. It follows that the third Taylor Polynomial is

$$\begin{aligned} f(t) &= f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \frac{f'''(0)}{3!}t^3 + \mathcal{O}\left(t^4\right) \\ \log(1+t) &= 0 + (1)t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \mathcal{O}\left(t^4\right) \\ \log(1+t) &= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \mathcal{O}\left(t^4\right) \end{aligned}$$

□

It follows that x_n is now

$$x_n = \log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \mathcal{O}\left(\frac{1}{n^4}\right) \tag{4}$$

We can now find the constants c and d as follows

$$
\begin{aligned}
z_n = \left(n + c + \frac{d}{n}\right) x_n &= \left(n + c + \frac{d}{n}\right) \log\left(1 + \frac{1}{n}\right) \\
&= n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&\quad + c\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&\quad + \frac{d}{n}\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= 1 - \frac{1}{2n} + \frac{1}{3n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \\
&\quad + \frac{c}{n} - \frac{c}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \\
&\quad + \frac{d}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \\
\hline
&= 1 + \left(c - \frac{1}{2}\right)\frac{1}{n} + \left(\frac{1}{3} - \frac{c}{2} + d\right)\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)
\end{aligned}
$$

By choosing $c = \dfrac{1}{2}$ and $d = -\dfrac{1}{12}$ we can speed up the convergence from $\mathcal{O}\left(\dfrac{1}{n^2}\right)$ to $\mathcal{O}\left(\dfrac{1}{n^3}\right)$. Therefore, the sequence

$$e_n^{(2)} = e^{z_n} = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}-\frac{1}{12n}}$$

converges to e even faster. In particular, they convege at these rates:

$$e_n = \exp\left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\},$$

$$e_n^{(1)} = \exp\left\{1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}, \text{ and}$$

$$e_n^{(2)} = \exp\left\{1 + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}$$

Summary

The table below illustrates the values of the sequences for

$$e_n = \left(1 + \frac{1}{n}\right)^n \qquad e_n^{(1)} = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \qquad e_n^{(2)} = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}-\frac{1}{12n}}$$

n	e_n	$e_n^{(1)}$	$e_n^{(1)}$
1	2.00000000000000	2.82842712474619	2.66967970834007
2	2.25000000000000	2.75567596063108	2.70951158243786
3	2.37037037037037	2.73706794282489	2.71528273178665
4	2.44140625000000	2.72957516784642	2.71691530287724
5	2.48832000000000	2.72581798858251	2.71754763748677
10	2.59374246010000	2.72034004202750	2.71818026568938
20	2.65329770514442	2.71882109520485	2.71826843585087
200	2.71151712292932	2.71828746336339	2.71828181438111
400	2.71489174438123	2.71828324069966	2.71828182669427
600	2.71602004888065	2.71828245664394	2.71828182793574
800	2.71658484668247	2.71828218196003	2.71828182823809
1000	$\underline{2.71}692393223559$	$\underline{2.71828}205475592$	$\underline{2.718281}82834561$

Table 1: Convergence of three sequences

The error for all of the sequences is

$$
\begin{aligned}
|e_n \quad - e| &\leqslant 0.00135789622345150 \\
|e_n^{(1)} - e| &\leqslant 0.00000226296876348897 \\
|e_n^{(2)} - e| &\leqslant 0.00000000113432818693582
\end{aligned}
$$

Conclusion

- This example is meant to help you understand what the convergence of a sequence means.
- It also shows you how Taylor polynomials are used.
- Knowing a sequence converges is useful.
- But knowing it converges fast is better!
- We will talk more in depth on convergence and rates of convergence in Chapter 2.