Math 311

Numerical Methods

2.2: Fixed Point Iteration $[f(x) = 0 \iff g(p) = p]$

Solutions of Equations in One Variable

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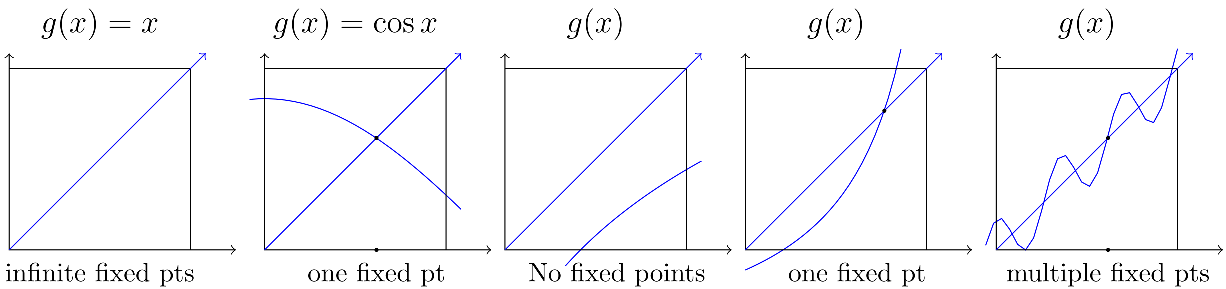
1 Introduction

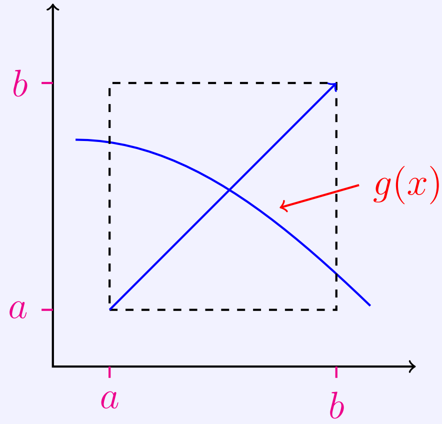
- An equivalent way of solving $f(x) = 0$ is to reformulate it as a fixed point problem.
- A function $g(x)$ has a fixed point at p if $g(p) = p$.
- Convert the problem of $f(x) = 0$ into $x = g(x)$ (solve for x—be creative!).
- To find "an" equivalent $g(x)$ for any $f(x)$, start with $f(x) = 0$ and solve for x in algebraic or sneaky methods. For example,
  - $f(x) = \cos x - x = 0$ is equivalent to $g(x) = \cos x = x$ (or $g(x) = \cos^{-1}(x)$)
  - $f(x) = x^2 - 2x + 3 = 0$ is equivalent to $g(x) = \frac{x^2+3}{2} = x$ (just one of many!).
- Picking the right $g(x)$ function can lead to powerful root finding techniques.
- Here are some examples of fixed points. When will it be unique?



Theorem: Uniqueness Conditions: (Thm 2.2)

- *If, for every* $x \in [a,b]$, $g(x) \in [a,b]$ *and is continuous, then* g *has a fixed point in* $[a,b]$.
- *Suppose further that* $\underline{g'(x) \text{ is defined on } (a,b)}$ *and that a positive constant* $k < 1$ *exists with*

$$|g'(x)| \leqslant k < 1, \text{ for all } x \in (a,b). \tag{1}$$



- *Then the fixed point in* $[a,b]$ *is unique.*

Think this: a good $g(x)$ *will enter on the "left wall" and exit on the "right wall".*

Proof. • First, we will show that a fixed point exists.

- If $g(a) = a$ or $g(b) = b$, then the fixed point exists automatically.
- So, then suppose that $g(a) \neq a$ and $g(b) \neq b$. It follows that $g(a) > a$ and $g(b) < b$.
- Define $h(x) = g(x) - x$. It follows that h is continuous on $[a,b]$ and

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0$$

- By the Intermediate Value Thm, there exists a $p \in (a, b)$ such that $h(p) = 0$.
- Thus, $h(p) = 0 = g(p) - p \Longrightarrow g(p) = p$.
- Thus the fixed point exists! Assume that that $|g'(x)| \leqslant k < 1$.
- Is the fixed point unique? We will suppose it isn't unique and show a contradiction occurs.
- Let's call the fixed points p and q, where $p \neq q$.
- By the Mean Value Theorem, there exists ξ between p and q with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi)$$

- Since $g(p) = p$ and $g(q) = q$, then it follows:

$$|p - q| = \underbrace{|g(p) - g(q)| = |g'(\xi)||p - q|}_{|\text{Mean Value Theorem}|} \leqslant k|p - q| < |p - q|$$

- So it follows that $|p - q| < |p - q| \Longrightarrow$ That's impossible!
- Therefore, the fixed point must be unique!

□

1.1 Example

1. Let $g(x) = \ln(7/x)$ on $[1, 2]$. It follows that $g'(x) = -\frac{1}{x}$.
   - Note that $g'(x) \neq 0$ and $g'(x) < 0$. This means g is 1-1 and decreasing.
   - By the EVT, the maximum of g will be at the endpoints of $[1, 2]$.
   - Since $g(1) = \ln(7) \approx 1.95$ and $g(2) = \ln(7/2) \approx 1.25$, it follows that

     $$g(x) \in [1.25, 1.95] \subset [1, 2], \implies g(x) \in [1, 2]$$

   - Thus, a fixed point exists in $[1, 2]$.
   - Next, we want to find a k such that $|g'(x)| \leqslant k < 1$ over $[1, 2]$.
   - $|g'(1)| = 1$ and $|g'(2) = \frac{1}{2}$, So $\max_{x\in[1,2]} |g'(x)| \leqslant 1$.
   - So does k exist? (k should be less than 1). Currently it doesn't exist.
   - However, it will exist if we shrink the interval some.
   - Suppose the interval is $[1.5, 2]$ instead.
   - Then $\max_{x\in[1,2]} |g'(x)| \leqslant |g'(1.5)| = \frac{2}{3} = k < 1$.
   - So the fixed point exists in the interval $[1.5, 2]$ and is unique!

2 How does it work?

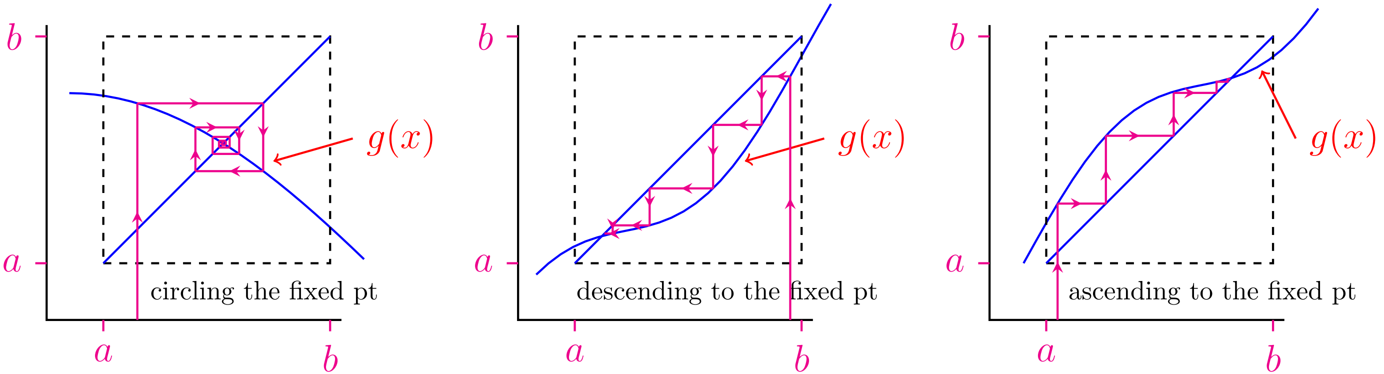
- To approximate a fixed pt, we choose an inital approx p_0
- Then generate a sequence $\{p_n\}_{n=0}^{\infty}$ by letting

$$p_n = g(p_{n-1}), \text{ for each } n \geqslant 1$$

- If the sequence converges and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\Big(\lim_{n\to\infty} p_{n-1}\Big) = g(p)$$

- How it looks visually is one of three cases:



3 How do you choose $g(p)$ for a particular $f(x) = 0$?

- Suppose we want to solve $x^3 + 4x^2 - 10 = 0.$
- By IVT, it has a root in $[1, 2]$
- Start with $f(x) = 0$ and then solve for x.

$$x^3 + 4x^2 - 10 = 0 \Longrightarrow 4x^2 = 10 - x^3 \qquad \text{(solve for } 4x^2\text{)}$$

$$x^2 = \frac{10 - x^3}{4} \qquad \text{(divide by 4)}$$

$$x = \frac{\sqrt{10 - x^3}}{2} \qquad \text{(choose positive square root)}$$

- Let's call this one $g_3(x) = \dfrac{\sqrt{10 - x^3}}{2}$. Now, let's find a new one.
- Since $f(x) = 0$, take $x - f(x)$ for another possible g:
- So $g_1(x) = x - f(x) = x - x^3 - 4x^2 + 10$. (Just another option to use!)
- We can choose others as well. Just solve for a different "x".

- Let's solve for the x^3.

$$x^3 + 4x^2 - 10 = 0 \Longrightarrow x^3 = 10 - 4x^2 \qquad \text{(Solve for } x^3\text{)}$$

$$x^2 = \frac{10 - x^3}{x} \qquad \text{(divide by } x\text{)}$$

$$x = \sqrt{\frac{10 - x^3}{x}} = g_2(x) \qquad \text{(square root both sides)}$$

- What else can we do? Be creative!

$$x^3 + 4x^2 - 10 = 0 \Longrightarrow x^3 + 4x^2 = 10 \qquad \text{(factor } x^2 \text{ on left)}$$

$$x^2(x + 4) = 10 \qquad \text{(divide by } x + 4\text{)}$$

$$x^2 = \frac{10}{x + 4} \qquad \text{(square root)}$$

$$x = \sqrt{\frac{10}{x + 4}} \Longrightarrow g_4(x) = \sqrt{\frac{10}{x + 4}}$$

- There are lots more ways! Some are very creative! Here's a crazy way to find one:

$$
\begin{aligned}
0 = x^3 + 4x^2 - 10 &= (3-2)x^3 + (8-4)x^2 - 10 \\
&= 3x^3 - 2x^3 + 8x^2 - 4x^2 - 10 \\
2x^3 + 4x^2 + 10 &= 3x^3 + 8x^2 \\
&= x(3x^2 + 8x) \\
\frac{2x^3 + 4x^2 + 10}{3x^2 + 8x} &= x
\end{aligned}
$$

- So we got $g_5(x) = \dfrac{2x^3 + 4x^2 + 10}{3x^2 + 8x}$.
- Let's try out all these methods and see how they perform.

4 How good are they? Let's test them!

- Let's start with $p_0 = 1$.

$$g_1(x) = x - x^3 - 4x^2 + 10 \qquad \text{(fails to converge)}$$

$$g_2(x) = \sqrt{\frac{10 - x^3}{x}} \qquad \text{(fails to converge)}$$

$$g_3(x) = \frac{\sqrt{10 - x^3}}{2} \qquad \text{(converges in 30 iterations)}$$

$$g_4(x) = \sqrt{\frac{10}{x + 4}} \qquad \text{(converges in 15 iterations)}$$

$$g_5(x) = \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x} \qquad \text{(converges in 4 iterations!)}$$

- Note that it is difficult to tell which converges by sight.
- How can we determine which will converge and how rapidly?
- We have Thm 2.3 in the book to help with this!

Theorem: Fixed point Theorem (Thm 2.3)

Let $g \in C[a,b]$, $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose $g'(x)$ exists on (a,b) and

$$|g'(x)| \leqslant k < 1 \text{ for all } x \in [a,b]$$

If p_0 is any number in $[a,b]$, then the sequence defined by

$$p_n = g(p_{n-1}), n \geqslant 1$$

converges to the unique fixed point p in $[a,b]$.

Proof.

- By Thm 2.2, a unique fixed point exists. Thus, the sequence $p_n \in [a,b]$.
- By the MVT, we know that there exists a $\xi \in (p, p_{n-1})$ (or $\xi \in (p_{n-1}, p)$) such that

$$\frac{g(p_{n-1}) - g(p)}{p_{n-1} - p} = g'(\xi).$$

- Also, remember that $p_n = g(p_{n-1})$ and $g(p) = p$. So it follows that

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)||p_{n-1} - p|$$

- Since $|g'(x)| \leqslant k < 1$, then it follows that

$$|p_n - p| \leqslant k|p_{n-1} - p| \tag{2}$$

- Note that (2) also means:

$$\begin{aligned} |p_{n-1} - p| &\leqslant k|p_{n-2} - p| \\ |p_{n-2} - p| &\leqslant k|p_{n-3} - p| \\ &\vdots \\ |p_1 - p| &\leqslant k|p_0 - p| \end{aligned} \tag{3}$$

- Combining (2) and (3) yields:

$$|p_n - p| \leqslant k|p_{n-1} - p| \leqslant k^2|p_{n-2} - p| \leqslant k^3|p_{n-3} - p| \leqslant \cdots \leqslant k^n|p_0 - p| \tag{4}$$

- Since $k < 1$, then $k^n \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} |p_n - p| \leqslant \lim_{n\to\infty} k^n|p_0 - p| = 0$
- Therefore p_n converges to p.

$\square$

Error Bound for Fixed Point

Corollary. *When g satisfies the conditions of Theorem 2.3, then we have the following bounds on the error:*

$$|p_n - p| \leqslant k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leqslant \frac{k^n |p_0 - p_1|}{1-k}, \text{ for all } n \geqslant 1$$

Proof. • Since $p_0 \in (a, b)$ and we don't know p, then it follows from (4) that

$$|p_n - p| \leqslant k^n |p_0 - p| < k^n \max\{p_0 - a, b - p_o\}$$

• We can get a better bound on this as follows. Suppose that $m > n \geqslant 1$. Then

$$\begin{aligned} |p_m - p_n| &= |p_m \underbrace{-p_{m-1} + p_{m-1} - p_{m-2} + p_{m-2} - p_{m-3} + \cdots - p_{n+1} + p_{n+1}}_{\text{equal to zero!}} - p_n| \\ &\leqslant |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + |p_{m-2} - p_{m-3}| + \cdots + |p_{n+1} - p_n| \\ &\leqslant k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + k^{m-3}|p_1 - p_0| + \cdots + k^n |p_1 - p_0| \\ &\leqslant k^n |p_1 - p_0| (1 + k + k^2 + \cdots + k^{m-n-1}) \qquad (5) \end{aligned}$$

- The right side of (5) is a geometric sum. Recall that the sum of a geometric series is

$$\sum_{i=0}^{b} k^i = \frac{1-k^{b+1}}{1-k},$$

so it follows that

$$1 + k + k^2 + \cdots + k^{m-n+1} = \frac{1-k^{m-n}}{1-k} \tag{6}$$

- Combining (5) and (6) yields

$$\begin{aligned} |p_m - p_n| &\leqslant k^n |p_1 - p_0| (1 + k + k^2 + \cdots + k^{m-n-1}) \\ &\leqslant k^n |p_1 - p_0| \frac{1-k^{m-n}}{1-k} \end{aligned} \tag{7}$$

Since $k^{m-n} \to 0$ and $p_m \to p$ as $m \to \infty$, then

$$\begin{aligned} \lim_{m\to\infty} |p_n - p_m| &\leqslant \lim_{m\to\infty} k^n |p_1 - p_0| \frac{1-k^{m-n}}{1-k} \\ |p_n - p| &\leqslant \frac{k^n |p_1 - p_0|}{1-k} \end{aligned} \tag{8}$$

- Note that the formula shows that the smaller the k, the faster the convergence.

□

Theorem: How many iterations for a specific value of k?

We can solve the equation in (8) for n. So, for a given value of k, the number of iterations to solve the equation to a specified tolerance (ε) is

$$n \geqslant \frac{\log\left(\frac{(1-k)\varepsilon}{|p_1-p_0|}\right)}{\log k} \qquad (9)$$

Proof.

- Let ε be the tolerance (e.g. value of ε such that $|p_n - p| \leqslant \varepsilon$).
- It follows that

$$|p_n - p| \leqslant \frac{k^n|p_1 - p_0|}{1-k} \leqslant \varepsilon \Longrightarrow k^n \leqslant \frac{(1-k)\varepsilon}{|p_1 - p_0|} \qquad \text{(Isolate } k^n\text{)}$$

$$n \log k \leqslant \log\left(\frac{(1-k)\varepsilon}{|p_1 - p_0|}\right) \qquad \text{(log of both sides)}$$

$$n \geqslant \frac{\log\left(\frac{(1-k)\varepsilon}{|p_1-p_0|}\right)}{\log k} \qquad \text{(divide by } \log k\text{)}$$

□

5 What is k for g_1, g_2, g_3, g_4, g_5 (to solve $f(x) = x^3 + 4x^2 - 10$)

- Analyzing with Desmos.com is a good strategy.
- However, it still comes down to using the Extreme Value Theorem.

$$g_1(x) = x - x^3 - 4x^2 + 10 \qquad (g'(x) \text{ is NEVER} < 1 \text{ over } [1,2])$$

$$g_2(x) = \sqrt{\frac{10 - x^3}{x}} \qquad (\text{Bad – Doesn't map [1,2] onto [1,2] } (g_2'(p) = 3.4 > 1))$$

$$g_3(x) = \frac{\sqrt{10 - x^3}}{2} \qquad (\text{[1,2] fails, but [1,1.5] works. } (g_3'(x) \leqslant \tfrac{2}{3}))$$

$$g_4(x) = \sqrt{\frac{10}{x + 4}} \qquad (g'(x) \leqslant \tfrac{\sqrt{2}}{10} \approx .14 < 1)$$

$$g_5(x) = \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x} \qquad (g'(p) = 0 < 1)$$

- These all correspond to the results from before!

- Analyzing g_5 a bit more will be the key to fast convergence (next section!).
- What does $g_5'(x)$ look like over [1,2]?
- $g_5'(x) = \dfrac{(6x+8)(x^3+4x^2-10)}{x^2(3x+8)^2}$
- The EVT says the max of $|g_5'(x)|$ will be at the endpoints or where $g''(x) = 0$.
- Since $g''(x) \neq 0$ for all $x \in [1,2]$, then just check both endpoints.
- $|g'(1)| = \frac{70}{121} = 0.579$ and $g'(2) = \frac{5}{14} \approx .357$. It follows that $k = 0.579$.
- Let $p_0 = 1$, and $\varepsilon = 10^{-8}$. It follows that $p_1 = \frac{16}{11}$ and using (9) will show

$$n \geqslant \frac{\log\left(\dfrac{\left(1-\dfrac{70}{121}\right)(10^{-8})}{\left|\dfrac{16}{11}-1\right|}\right)}{\log\left(\dfrac{70}{121}\right)} = 33.7$$

- **This problem converges way faster than this.**
- **In fact, only 4 are needed! Why?**

- What happens near the fixed point? As you can see from above $g_5'(p) = 0$
- As we shrink the interval, the value of k changes.
- As the interval collapses around p, k gets closer and closer zero!
- Remember a small k value leads to fast convergence!
- Choose a different starting point and the calculations change!
- Suppose you start at $p_0 = \frac{4}{3}$. Then $p_1 = \frac{295}{216}$. At p_0, $|g'(p_0)| = \frac{7}{216}$.
- So we get this time:

$$n \geqslant \frac{\log\left(\dfrac{\left(1 - \frac{7}{216}\right)(10^{-8})}{\left|\frac{295}{216} - \frac{4}{3}\right|}\right)}{\log\left(\frac{7}{216}\right)} = 4.36$$

- We will now discuss the method we just illustrated – The Newton-Raphson Method.
- There is a Desmos assignment on Canvas that you need to complete! Go find it!