Math 311

Numerical Methods

2.4: Error Analysis for Iterative Methods

Solutions of Equations of One Variable

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1 Introduction

- Suppose we have a sequence of real numbers $\{p_n\}_{n=0}^{\infty}$ that converge to a number p.
- In other words, $p_n \to p$.
- We want to talk about how fast the values are converging to p as n increases.
- Allowing us to talk about this enables us to make improvements and progress!
- We can then develop new methods and compare them to the older methods.
- We will be discussing the limit in the box below. Note that the top and bottom are how close the iterations are to p. Note also the bottom has a power of a postive α.

$$\lim_{n\to\infty} \frac{|\, p_{n+1} - p \,|}{|\, p_n - p \,|^{\alpha}}, = \; ? \tag{1}$$

- The answer to this limit will tell us the speed of a sequence.
- What values are possible? (Discuss!)

$\to \infty$ (bad news).

$\to 0$ (good news)

$\to$ somewhere in between $0 < \lambda < \infty$ (finite and positive) (best news)

Convergence a sequence of order α.

Definition. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a sequence that converges to* p*.*
If constants $\lambda > 0$ *and* $\alpha > 0$ *exist with*

$$0 < \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}}, = \lambda < \infty, \quad \text{then} \tag{2}$$

$\{p_n\}_{n=0}^{\infty}$ ***converges to p of order*** α, *with* ***asymptotic error constant*** λ.

Convergence examples

Definition. *Further vocabulary on speeds of convergence include:*

- *If* $\alpha = 1$ *and* $\lambda > 0$ *then the sequence is said to converge* ***linearly*** *to* p.
- *If* $\alpha = 2$ *and* $\lambda > 0$ *then the sequence is said to converge* ***quadratically*** *to* p.
- *If* $\alpha = 3$ *and* $\lambda > 0$ *then the sequence is said to converge* ***cubically*** *to* p.
- *Similar language is used for larger values of* α

Corollary. *If* $0 < \alpha < 1$ *and* $\lambda > 0$, *the sequence converges* ***sub-linearly*** *to* p.

Super-convergence of a sequence of order α.

Definition. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a sequence that converges to* p*.*
If a positive constant $\boldsymbol{\alpha > 0}$ *exists with*

$$\lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\alpha}} = \mathbf{0}, \quad \text{then} \tag{3}$$

- $\{p_n\}_{n=0}^{\infty}$ *is said to converge to* $\mathbf{p}$ *of order* ***super-α*** *.*
- *(or, equivalently* p_n ***converges super-α-ly to p****).*
- *(Note, the asymptotic error constant here MUST be* $\boldsymbol{\lambda = 0}$*).*

Super-convergence examples

Definition. *Further vocabulary on speeds of convergence include:*

- *If* $\alpha = 1$ *and* $\lambda = 0$ *then* p_n *is said to* ***superlinearly*** *converge to* p*.*
- *If* $\alpha = 2$ *and* $\lambda = 0$ *then* p_n *is said to* ***super-quadratically*** *converge to* p*.*
- *If* $\alpha = 3$ *and* $\lambda = 0$ *then* p_n *is said to* ***super-cubically*** *converge to* p*.*
- *If* $\alpha = 4$ *and* $\lambda = 0$ *then* p_n *is said to* ***super-quartically*** *converge to* p*, etc.*

Super-α-convergence implies existence of convergence of order β

Theorem. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *converges super-*α*-ly to* p*. It follows that:*

- *There exists* $\beta > \alpha$ *such that the asymptotic error constant,* λ_β*, is finite and positive.* $(0 < \lambda_\beta < \infty)$
- *In other words,* ***there exists a constant*** β ***larger than*** α ***such that*** p_n *converges to* p *of order* β

Example.

- *The Secant method is known for its superlinear (super-1-ly) convergence.*
- *This implies that it converges at a rate greater than 1.*
- *In fact, that rate is* $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$.
- *This means that once it gets close enough, it will improve upon the last iteration by increasing the number of digits correct by 61.8%.*

Convergence of order α implies divergence for all $\beta > \alpha$

Theorem. *Suppose that* $\boldsymbol{p_n}$ *converges to* $\boldsymbol{p}$ *of order* $\boldsymbol{\alpha}$ *with* $\lambda > 0$.

For every $\beta > \alpha$, $\boldsymbol{p_n}$ *does not converge to* $\boldsymbol{p}$ *of order* $\boldsymbol{\beta}$. $\left(\lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\beta}} = \infty\right)$

Proof. Let $\epsilon > 0$. Suppose $\beta = \alpha + \epsilon$. It follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\beta}} &= \lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\alpha+\epsilon}} = \lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\alpha}} \left(\frac{1}{|p_n - p|^{\epsilon}}\right) \\
&= \lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\alpha}} \lim_{n\to\infty} \left(\frac{1}{|p_n - p|^{\epsilon}}\right) \\
&= \lambda \cdot \lim_{n\to\infty} \left(\frac{1}{|p_n - p|^{\epsilon}}\right) = \infty
\end{aligned}$$

Therefore, it follows that $\{p_n\}_{n=0}^{\infty}$ does not converge to p of order $\boldsymbol{\beta > \alpha}$. □

Convergence of order α implies superconvergence for all $\beta < \alpha$

Theorem. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *converges to* p *of order* α. *It follows that:*

For every $\beta < \alpha$, p_n *converges to* p *of order super-*β.

Proof. Let $\epsilon > 0$. Suppose $\beta = \alpha - \epsilon$. It follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\beta}} &= \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha-\epsilon}} = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} \left(|p_n - p|^{\epsilon}\right) \\
&= \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} \lim_{n\to\infty} \left(|p_n - p|^{\epsilon}\right) \\
&= \lambda \cdot \lim_{n\to\infty} \left(|p_n - p|^{\epsilon}\right) = 0
\end{aligned}$$

Therefore, it follows that p_n converges to p of order super-β, where $\beta = \alpha - \epsilon$. □

- So, in summary, for every sequence that converges of order α, we have one of the following three options:
  - For α, the sequence p_n converges to p of order α.
  - For all values less than α, the seqence converges super-α-ly to p.
  - For all values greater than α, the seqence does not converges to p.

1.1 Example 1

1. Let $p_n = \frac{1}{n^k}$ for some fixed $k > 0$. This sequence converges to $p = 0$. By the definition in (2),

$$\lambda = \lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^{\alpha}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\left[\frac{1}{n^k}\right]^{\alpha}} = \lim_{n\to\infty} \left[\frac{n^{\alpha}}{n+1}\right]^k = \left[\lim_{n\to\infty} \frac{n^{\alpha-1}}{1+\frac{1}{n}}\right]^k = \left[\lim_{n\to\infty} n^{\alpha-1}\right]^k$$

It follows that

- If $\alpha < 1$ then $\lambda = 0$, which implies super-α convergence.
- If $\alpha = 1$ then $\lambda = 1$, which implies the sequence converges linearly.
- If $\alpha > 1$ then $\lambda = \infty$, which implies it does not converge at any rate greater than linear.

1.2 Example 2

2. Let $p_n = 10^{-2^n}$. This also converges to 0 and the asymptotic error constant is

$$\lambda = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-\alpha\cdot 2^n}} = \lim_{n\to\infty} 10^{-2\cdot 2^n} \cdot 10^{\alpha\cdot 2^n} = \lim_{n\to\infty} 10^{(\alpha-2)\cdot 2^n}$$

- If $\alpha < 2$, then $\lambda = 0$. This means that p_n converges super-linearly to p.
- If $\alpha = 2$, then $\lambda = 1$. This means that p_n converges quadratically to p.
- If $\alpha > 2$, then $\lambda = \infty$ and p_n does not converge at a rate larger than $\alpha = 2$.

Using this as a typical quadratic sequence, you can see how fast a quadratically convergent sequence moves. The terms of the sequence are:

n	p_n
1	10^{-2}
2	10^{-4}
3	10^{-8}
4	10^{-16}
5	10^{-32}
6	10^{-64}

It doubles the number of correct digits with each iteration! Note that after 6 iterations, it is already really small!

2 Fixed Point Iteration (Section 2.2)

How good is fixed point iteration? Let's analyze the fixed point algorithm,

$$p_{n+1} = g(p_n)$$

with fixed point p. The key to the speed of convergence is derivatives of $g(p)$.

Convergence of Fixed Point Iteration:

Theorem. *Let $g \in C[a,b]$ and $g' \in C(a,b)$. Furthermore, assume there exists* $k < 1$ *such that* $|g'(x)| \leqslant k$ *for all x in (a,b).*

- *If $g'(p) \neq 0$, the sequence converges linearly to the fixed point p.*
- *If $g'(p) = 0$, the sequence converges at least quadratically to the fixed point p.*

Proof. • First, we will show that $p_n \to p$. Start with the statement

$$p_n = g(p_{n-1})$$

- Subtract p from both sides and take the absolute value:

$$|p_n - p| = |g(p_{n-1}) - p|$$

- Note that since $g(\boldsymbol{p}) = p$, it follows that

$$|\boldsymbol{p_n} - \boldsymbol{p}| = |g(p_{n-1}) - g(\boldsymbol{p})|$$

- Suppose that $|g'(x)| \leqslant k < 1$. By the MVT, ξ exists between p_{n-1} and $\boldsymbol{p}$ where

$$\underbrace{|\boldsymbol{p_n} - \boldsymbol{p}|}_{\text{left side}} = |g(p_{n-1}) - g(\boldsymbol{p})| = |g'(\xi)||p_{n-1} - p| \leqslant \underbrace{k|p_{n-1} - p|}_{\text{right side}}$$

- Thus, the left side is less than or equal to the right side!

$$|\boldsymbol{p_n} - \boldsymbol{p}| \leqslant k|p_{n-1} - \boldsymbol{p}|. \tag{4}$$

- Recursively plugging equation (4) into itself yields:

$$|\boldsymbol{p_n} - \boldsymbol{p}| \leqslant k|k|p_{n-2} - \boldsymbol{p}|| \leqslant k|k|k|p_{n-3} - \boldsymbol{p}||| \leqslant \cdots \leqslant k^n|p_0 - p|.$$

- Since $0 \leqslant k < 1$, then $k^n \to 0$ as $n \to \infty$.
- Therefore, $|\boldsymbol{p_n} - \boldsymbol{p}| \to 0$ as $n \to \infty$ (which is equivalent to $\boldsymbol{p_n} \to p$).
- Next, it can be shown that convergence speed will depend on the derivatives of g.
- By the Mean Value Theorem, ξ exists between p_{n-1} and $\boldsymbol{p}$ where

$$|\boldsymbol{p_{n+1}} - \boldsymbol{p}| = |g(p_n) - g(\boldsymbol{p})| = |g'(\xi_n)||p_n - p|$$

- Since $\boldsymbol{p_n} \to p$, then $\xi_n \to p$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{|\boldsymbol{p_{n+1}} - \boldsymbol{p}|}{|\boldsymbol{p_n} - \boldsymbol{p}|} = \lim_{n\to\infty} |g'(\xi_n)| = \left|g'\left(\lim_{n\to\infty} \xi_n\right)\right| = |g'(\boldsymbol{p})|$$

- It follows that

$$\lim_{n\to\infty} \frac{|\boldsymbol{p_{n+1}} - \boldsymbol{p}|}{|\boldsymbol{p_n} - \boldsymbol{p}|} = |g'(\boldsymbol{p})| \tag{5}$$

- By the definition of convergence and super-convergence in (2) and (3), we see that:

$$\begin{cases} \text{If } |g'(\boldsymbol{p})| \neq 0, \text{ then } \boldsymbol{p_n} \text{ converges to } \boldsymbol{p} \text{ at a linear rate } (\lambda = |g'(\boldsymbol{p})| > 0). \\ \text{If } |g'(\boldsymbol{p})| = 0, \text{ then } \boldsymbol{p_n} \text{ converges to } \boldsymbol{p} \text{ at a } \textbf{super-linear} \text{ rate.} \end{cases}$$

- To precisely find the order that $\boldsymbol{p_n}$ converges to $\boldsymbol{p}$, expand $g(x)$ in a Taylor's Polynomial about $\boldsymbol{p}$. It follows that by Taylor's Theorem,

$$g(x) = g(\boldsymbol{p}) + g'(\boldsymbol{p})(x - \boldsymbol{p}) + \frac{1}{2}g''(\xi)(x - \boldsymbol{p})^2, \tag{6}$$

where ξ is between x and $\boldsymbol{p}$.

- Since $g(p) = p$ and $g'(p) = 0$, then it follows that (6) simplifies as

$$g(x) = g(p) + \cancelto{0}{g'(p)}(x - p) + \frac{1}{2}g''(\xi)(x - p)^2$$

$$g(x) = p + \frac{1}{2}g''(\xi)(x - p)^2 \tag{7}$$

- Evaluate (7) at $x = p_n$. Note when $x = p_n$, $g(p_n) = p_{n+1}$,

$$g(p_n) = p + \frac{1}{2}g''(\xi_n)(p_n - p)^2$$

$$g(p_n) - p = \frac{1}{2}g''(\xi_n)(p_n - p)^2$$

$$\frac{|\, p_{n+1} - p\,|}{|\, p_n - p\,|^2} = \frac{1}{2}|g''(\xi_n)|$$

- p_n converges to p. Further, since ξ_n is always between p_n and p, then ξ_n converges to p as well. It follows that

$$\lim_{n\to\infty} \frac{|\, p_{n+1} - p\,|}{|\, p_n - p\,|^2} = \frac{1}{2} \lim_{n\to\infty} |g''(\xi_n)| = \frac{1}{2}\left|g''\left(\lim_{n\to\infty} \xi_n\right)\right| = \frac{1}{2}|g''(p)|$$

- This yields a similar statement to (5).

$$\lim_{n\to\infty} \frac{|\,p_{n+1} - p\,|}{|\,p_n - p\,|^2} = \frac{1}{2}|g''(p)|$$

- We then can conclude that

$$\begin{cases} \text{If } |g''(p)| > 0, \text{ then } p_n \text{ converges to } p \text{ of order 2 (quadratically).} \\ \text{If } g''(p) = 0, \quad \text{then } p_n \text{ converges to } p \text{ at a super-quadratic rate (at least quadratic).} \end{cases}$$

□

2.1 Final Thoughts

- A side note: a key to finding faster fixed point methods is to generate a g for which $g'(p) = 0$. This leads to a quadratic rate.
- We can continue this idea and develop faster methods by finding a function $g(x)$ for which $g'(p) = g''(p) = 0$, but $g'''(p) \neq 0$. This will lead to cubic convergence rate.
- There are some examples of these methods, one called "Halley's Method" and another "Olvert's Method". We get a faster speed by in exchange with more complexity. There is also a series of methods called "Householder's Methods" which can generate sequences of any desired rate.

3 Newton's Method

- We will develop a faster fixed point method using the tricks above.
- We want to find $g(x)$ such that $g'(\boldsymbol{p}) = 0$, where $\boldsymbol{p}$ is the fixed point.
- In the past, to solve $f(x) = 0$, we created $g(x) = x - f(x)$, or something similar.
- Let's assume that the form of $g(x)$ is as follows and that we want to find $\phi(x)$ which forces $g'(\boldsymbol{p}) = 0$.

$$g(x) = x - \phi(x)f(x)$$
$$g'(x) = 1 - \phi'(x)f(x) - \phi(x)f'(x) \qquad \text{(Using the product rule)}$$

- Since $f(\boldsymbol{p}) = 0$ and forcing $g'(\boldsymbol{p}) = 0$, it follows that

$$0 = g'(\boldsymbol{p}) = 1 - \phi'(\boldsymbol{p})\cancel{f(\boldsymbol{p})} - \phi(\boldsymbol{p})f'(\boldsymbol{p}) \qquad \Longrightarrow \qquad \phi(\boldsymbol{p}) = \frac{1}{f'(\boldsymbol{p})}$$

- Therefore, the function to iterate is:

$$g(x) = x - \phi(x)f(x) = x - \left(\frac{1}{f'(\boldsymbol{p})}\right)f(x)$$
$$g(x) = x - \frac{f(x)}{f'(\boldsymbol{p})}$$

- This is VERY similar to Newton's method, except we don't know $f'(\boldsymbol{p})$.
- Since $\boldsymbol{p}$ is usually unknown, then let $\boldsymbol{p} = \boldsymbol{p_n}$ and then

$$\boldsymbol{p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}},$$

which IS Newton's Method.

- We can show quadratic convergence by analyzing the derivative of $g(x)$ at $\boldsymbol{p}$.
- Taking the derivative of this function $g(x)$ is

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2} \tag{8}$$

- Since $f(\boldsymbol{p}) = 0$, then it is clear that $g'(\boldsymbol{p}) = 0$. Thus, Newton's method will converge at least quadratically!
- However, if $\boldsymbol{f'(p) = 0}$ at the same time as $g'(\boldsymbol{p})$, then we might not have quadratic convergence.

3.1 Problems with Newton's Method

- Here's an example of the problem:
  Let $f(x) = x^2$, so $f'(x) = 2x$. This has the the obvious solution of 0. So $g(x)$ is

$$g(x) = x - \frac{x^2}{2x} \Longrightarrow g(x) = \frac{x}{2}$$

- Note that $g'(x) = \frac{1}{2} \neq 0$
- All this method does it repeatedly half the answer from what was there before. This is clearly a linearly converging sequence. (Sounds like the Bisection)
- Why did this fail to have quadratic convergence?

Zero of Multiplicity m

Definition. *A solution* $\boldsymbol{p}$ *of* $f(x) = 0$ *is said to be a zero of multiplicity* m *of* f *if* $f(x)$ *can be written as*

$$f(x) = (x - \boldsymbol{p})^m q(x), \text{ for } x \neq p, \text{ where } \lim_{x \to p} q(x) \neq 0.$$

Zero of Multiplicity m (Part 2)

Theorem.
The function $f \in C^m[a,b]$ has a zero of multiplicity m at $\boldsymbol{p}$ if and only if

$$f(\boldsymbol{p}) = f'(\boldsymbol{p}) = f''(\boldsymbol{p}) = \cdots = f^{(m-1)}(\boldsymbol{p}) = 0, \qquad \textit{but} \qquad f^{(m)}(\boldsymbol{p}) \neq 0.$$

3.2 Example

The function $f(x) = 2\cos x - 2 - x^2$ has a zero of multiplicity 2 at $x = 0$. Here is why:

$$\begin{aligned}
f(x) &= 2\cos x - 2 - x^2 & \Longrightarrow \quad f(0) &= 2(1) - 2 - 0^2 &&= 0\\
f'(x) &= -2\sin x - 2x & \Longrightarrow \quad f'(0) &= -2(0) - 2(0) &&= 0\\
f''(x) &= -2\cos x - 2 & \Longrightarrow \quad f''(0) &= -2(1) - 2 &&= -4 \neq 0
\end{aligned}$$

- When the zero is NOT simple, then Newton's method will converge <u>linearly</u>.

4 Modified Newton's Method

- What can we do to fix this problem, if possible?
- There are two methods to fix this:
  - $g(x) = x - \dfrac{mf(x)}{f'(x)}$. (Problem 8 in Section 2.4, 5th Ed.)
  - Apply Newton's Method to $\mu(x) = \dfrac{f(x)}{f'(x)}$.
- The first method requires knowledge of the multiplicity of the root. This information is not available in general.
- The second method has the advantage of not requiring knowledge of m.

- Let's focus on the second method.

- Suppose $f(x) = (x - \boldsymbol{p})^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.
- It follows that $f'(x) = m(x - \boldsymbol{p})^{m-1} q(x) + (x - \boldsymbol{p})^m q'(x)$
- The new function $\mu(x)$ simplifies to

$$\begin{aligned}
\mu(x) = \frac{f(x)}{f'(x)} &= \frac{(x - \boldsymbol{p})^m q(x)}{m(x - \boldsymbol{p})^{m-1} q(x) + (x - \boldsymbol{p})^m q'(x)} \\
&= \frac{(x - \boldsymbol{p})^m q(x)}{(x - \boldsymbol{p})^{m-1} [m q(x) + (x - \boldsymbol{p}) q'(x)]} && \text{(Factor out } (x - \boldsymbol{p})^{m-1}\text{)} \\
&= \frac{(x - \boldsymbol{p}) q(x)}{m q(x) + (x - \boldsymbol{p}) q'(x)} && \text{(Cancel factor of } (x - \boldsymbol{p})^{m-1}\text{)}
\end{aligned}$$

- What remains is a simple root of $\mu(x)$ at $x = p$.
- So apply Newton's to $\mu(x) = \dfrac{f(x)}{f'(x)}$!
- We will now ignore the form of $f(x)$ above, and keep it general.

- So it follows

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{\frac{f(x)}{f'(x)}}{\frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right)}$$

$$= x - \frac{\frac{f(x)}{f'(x)}}{\frac{f'(x)\cdot f'(x) - f(x)f''(x)}{[f'(x)]^2}}$$

$$= x - \frac{f(x)}{f'(x)}\left(\frac{[f'(x)]^2}{f'(x)\cdot f'(x) - f(x)f''(x)}\right)$$

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - [f(x)][f''(x)]} \tag{9}$$

- It follows that the "Modified Newton's Method" is

$$\boldsymbol{p_{n+1}} = \boldsymbol{p_n} - \frac{f(\boldsymbol{p_n})f'(\boldsymbol{p_n})}{[f'(\boldsymbol{p_n})]^2 - [f(\boldsymbol{p_n})][f''(\boldsymbol{p_n})]} \tag{10}$$

4.1 Example 1 of Modified Newton's Method

- The previous example of a slow down was applying Newton's Method to $f(x) = x^2$, which lead to $g(x) = \frac{1}{2}x$.

- Let's try the modified procedure on this function now.
- So $f'(x) = 2x$ and $f''(x) = 2$. The Modified Newton's simplifies to

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - [f(x)][f''(x)]} = x - \frac{x^2(2x)}{[2x]^2 - [x^2][2]}$$
$$= x - \frac{2x^3}{4x^2 - 2x^2} = x - \frac{2x^3}{2x^2} = x - x = 0$$

- Woah! That is really sped up! The guess is always 0! It found the root in 1 step!
- Any function like $f(x) = (x-c)^2$ will converge in 1 iteration to $x = c$ using the modified Newton's Method.

4.2 Example 2 of Modified Newton's Method

- Apply it to a similar problem: $f(x) = x^3 - 3x + 2$ has a double root at $x = 1$
- Newton's Method applied to it is $g(x) = x - \dfrac{x^3 - 3x + 2}{3x^2 - 3}$.
- With an initial guess of $x_0 = 2$, it takes 35 iterations to converge within 10^{-13}.
- The Modified Newton's simplifies to ($f'(x) = 3x^2 - 3$ and $f''(x) = 6x$)

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - [f(x)][f''(x)]} = x - \frac{(x^3 - 3x + 2)(3x^2 - 3)}{[3x^2 - 3]^2 - [x^3 - 3x + 2][6x]} = \frac{4x + 2}{x^2 + 2x + 3}$$

- With an initial guess of $x_0 = 2$, it takes 4 iterations to converge within 10^{-10}.

4.3 Example with $f(x) = 2\cos x - 2 - x^2$

- In example 2.3, we used $f(x) = 2\cos x - 2 - x^2$.
- It had a zero of multiplicity 2 at $x = 0$.
- Starting with a guess of $x_0 = 1$, we applied Newton's and Modified Newton's to it.
- It takes 37 iterations to converge within 10^{-8} with Newton's.
- It takes 7 iterations to converge within 10^{-8} with Modified Newton's.

4.4 Orders of Common Methods

Method	Iteration Formula ($p_{n+1} =$) or Combination	Worst Order	Best Order	Global Convergence?
Bisection	Not iteration	1	1	Yes
Fixed Point	$p_{n+1} = g(p_n)$	ϵ	–	No
Newton's	$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}$	1	2	No
Steffensen's	Fixed Pt & Aitkens	2	2	No
Modified Newton's I	$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)f'(p_n)}{[f'(p_n)]^2 - f(p_n)f''(p_n)}$	2	2	No
Modified Newton's II	$g(p_n) = p_n - m\frac{f(p_n)}{f'(p_n)}$	2	2	No
Secant	$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$	–	1.618	No
False Position	hybrid	1	1.618	Yes
Illinois	hybrid	–	1.442	Yes
Halley's Method	$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)f'(p_n)}{[f'(p_n)]^2 - \frac{1}{2}f(p_n)f''(p_n)}$	–	3	No

4.5 Other Methods we didn't cover

Method	Iteration Formula ($p_{n+1} =$) or Combination	Worst Order	Best Order	Global Convergence?
Brent's	hybrid	1.618	1.839	Yes
IQI	Inverse Quadratic Interpolation		1.839	No
ITP Method	Interpolate, Truncate, and Project	1	> 1	Yes
Mueller's Method	secant & IQI		1.839	?
Laguerre's Method	general poly root solver	1	3	Almost
Jenkins-Traub Method	complete polynomial root solver	1	2.618	Yes
Ridder's Method	false position variant	1.414	2	Yes
Durand-Kerner Method	simultaneously all roots of polynomial	1	2	Yes
Aberth Method	simultaneously all roots of polynomial	1	3	Yes