Math 311

Numerical Methods

4.3: Elements of Numerical Integration

Closed and Open Newton–Cotes

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Introduction

Numerical Quadrature

Numerical integration, also called numerical quadrature are used to evaluate a definite integral of a function that has no explicit antiderivative (or it's just difficult). To make the notation below more compact, define $\boldsymbol{y_k = f(x_k)}$. The main idea is to approximate it as a linear combination of a_k and y_k.

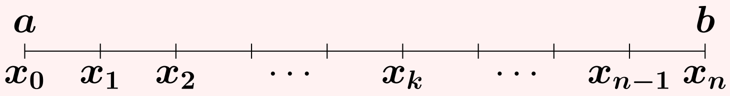
$$\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) = \sum_{k=0}^{n} a_k y_k,$$

Closed Newton-Cotes Formulas

- Typically, the integral of any function evaluates the anti-derivative of that function at the endpoints.

- Split the interval $[a, b]$ evenly into n intervals with width $h = \dfrac{b-a}{n}$.

Partition of Interval



$$\boldsymbol{x_0 = a,}$$
$$\boldsymbol{x_k = x_0 + kh}$$
$$\boldsymbol{x_n = b}$$

- It's called "closed" since the endpoints x_0 and x_n are used in the calculation of the integral.

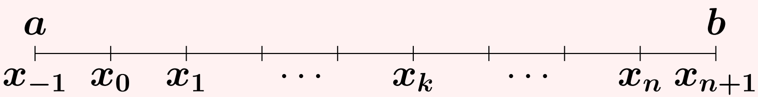
- The coefficients a_k are derived by using a Lagrange polynomial of degree n which fits the function at each of the node points and then we integrate the formula.

n	Name	Formula
1	Trapezoid rule	$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\left[y_0 + y_1\right] - \frac{h^3}{12} f''(\xi)$
2	Simpson's rule	$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] - \frac{h^5}{90} f^{(4)}(\xi)$
3	Simpson's three-eighths rule	$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right] - \frac{3h^5}{80} f^{(4)}(\xi)$
4	Boole's rule	$\int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right] - \frac{8h^7}{945} f^{(6)}(\xi)$
5		$\int_{x_0}^{x_5} f(x)dx = \frac{5h}{288}\left[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5\right] - \frac{275h^7}{12096} f^{(6)}(\xi)$
6		$\int_{x_0}^{x_6} f(x)dx = \frac{h}{140}\left[41y_0 + 216y_1 + 27y_2 + 272y_3 + 27y_4 + 216y_5 + 41y_6\right] - \frac{9h^9}{1400} f^{(8)}(\xi)$

Open Newton-Cotes Formulas

- Sometimes it is desirable to not evaluate the function at the end points. (For example, when we have a discontinuity at the endpoints).
- Split the interval $[a, b]$ is split evenly into $n+2$ intervals with width $h = \dfrac{b-a}{n+2}$.

Partition of Interval



$$\begin{aligned} x_{-1} &= a \\ x_0 &= a + h, \\ x_k &= x_0 + kh \\ x_n &= b - h \\ x_{n+1} &= b \end{aligned}$$

- It's called "open" since the endpoints a and b are NOT used in the calculation of the integral ($a = x_{-1}$ and $b = x_{n+1}$ are not used).
- The coefficients a_k are derived by using a Lagrange polynomial of degree n which fits the function at each of the node points.
- Here are the answers for the first few values of n, where $\xi \in (x_0, x_n)$.

n	Name	Formula
0	Midpoint rule	$\int_{x_{-1}}^{x_1} f(x)dx = 2h\ y_0 + \frac{h^3}{3} f''(\xi)$
1		$\int_{x_{-1}}^{x_2} f(x)dx = \frac{3h}{2}\ [y_0 + y_1] - \frac{3h^3}{4} f''(\xi)$
2	Milne's rule	$\int_{x_{-1}}^{x_3} f(x)dx = \frac{4h}{3}\ [2y_0 - y_1 + 2y_2] - \frac{14}{45} h^5 f^{(4)}(\xi)$
3		$\int_{x_{-1}}^{x_4} f(x)dx = \frac{5h}{24}\ [11y_0 + y_1 + y_2 + 11y_3] - \frac{95}{144} h^5 f^{(4)}(\xi)$
4		$\int_{x_{-1}}^{x_5} f(x)dx = \frac{3h}{10}\ [11y_0 - 14y_1 + 26y_2 - 14y_3 + 11y_4] - \frac{41}{40} h^7 f^{(6)}(\xi)$
5		$\int_{x_{-1}}^{x_6} f(x)dx = \frac{7h}{1440}[611y_0 - 453y_1 + 562y_2 + 562y_3 - 453y_4 + 611y_5] - \frac{5257}{8640} h^7 f^{(6)}(\xi)$