Math 311

Numerical Methods

4.4: Composite Numerical Integration

Traditional Midpoint, Trapezoid, and Simpsons Methods

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Introduction

Numerical Quadrature

Numerical integration, also called numerical quadrature are used to evaluate a definite integral of a function that has no explicit antiderivative (or it's just difficult). To make the notation below more compact, define $y_k = f(x_k)$. The main idea is to approximate it as a linear combination of a_k and y_k.

$$\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) = \sum_{k=0}^{n} a_k y_k,$$

1 Composite Closed Newton-Cotes Formulas

What can we do to increase the accuracy of these methods? What about partitioning the interval from a to b into more intervals, and applying the simple rules repeatedly. For example, the simple Simpsons Rule in section 1 requires 2 intervals (and 3 points). If we increase the number of intervals to 6 (7 points), then we can apply the simple Simpson's Rule to all three intervals and add up the result. More specifically, if the interval is split like so:

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

then the Composite Simpson's Rule would be:

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] + \frac{h}{3}\left[y_4 + 4y_5 + y_6\right] \\ &= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right]\end{aligned}$$

We can also calculate the integral using the formula corresponding to $n = 6$ in the Closed Newton-Cotes Formulas in section 1. What's the advantage of this over another Newton Cote's Rule that has more points? First, it is easily expandable. As the number of intervals increases, the h shrinks, which gives us smaller errors pretty fast. Second, as we increase the number of points, we will get wildly oscillating polynomials, which will be unstable.

The main drawback for using the composite rules is that the error is compounded each time we use the method. For instance, for the problem above, we used Simpson's Rule three times, which means the error should be 3 times what it was before. In general, we lose an order of magnitude on h by using the composite rule.

Name	Formula
Trapezoid rule	$\int_a^b f(x)dx = \frac{h}{2}\left[y_0 + y_n + 2\sum_{k=1}^{n-1} y_k\right] - \frac{(b-a)h^2}{12} f''(\xi)$
Simpson's rule	$\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + y_n + 4\sum_{k=1}^{(n/2)-1} y_{2k} + 2\sum_{k=1}^{n/2} y_{2k-1}\right] - \frac{(b-a)h^4}{180} f^{(4)}(\xi)$

Please note that while rules can be created for larger order $n \geqslant 3$, it is not typically done, as you can manipulate n in Simpson's Rule to get a similar result anyway.

2 Composite Open Newton-Cotes Formulas

Similar to the Closed Newton-Cotes Formula, we can create composite rules to increase our accuacy. We split the interval like the open methods above.

Name	Formula
Midpoint rule	$\int_a^b f(x)dx = 2h\sum_{k=0}^{n/2} y_{2k} + \frac{(b-a)h^2}{6} f''(\xi)$