$\boxed{2.2}$

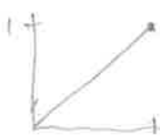
Fixed-Point Iteration

A fixed point for a function g is a number p for which $g(p)=p$

Root finding is equivalent: $\underbrace{g(p)-p}_{f(p)}=0$

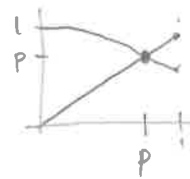
Picking the right $g(p)$ function for a given $f(x)$ can sometimes lead to powerful root finding techniques.

Ex: $g(x)=x$

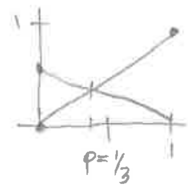


infinite fixed pts

$g(x)=\cos x - x$



$g(x)=\frac{-1}{2}x+\frac{1}{2}$



$g(x)=$

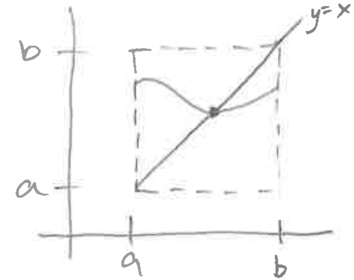
No unique one

How do we get a unique one over any particular $[a,b]$?

Sufficient conditions for uniqueness & existence of a fixed pt!

$\boxed{\text{Thm 2.1}}$ If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$ (both x & y are in the box)

then g has a fixed point in $[a,b]$



Suppose that $g \in C'(a,b)$ and

$\exists$ $0<k<1$ with

$$|g'(x)| \leq k < 1 \quad \forall\, x \in (a,b)$$

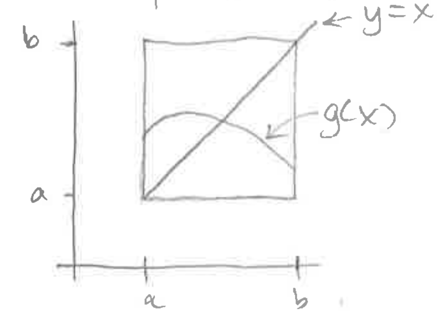
Then the fixed pt is unique in $[a,b]$

starts at left "wall" and ends at right "wall" and <u>slope</u> isn't too big!

<u>proof</u>: Next

Proof: If $g(a)=a$ or $g(b)=b$, then the fixed pt exists.

Suppose $g(a)\neq a$ and $g(b)\neq b$, then
$g(a)>a$ and $g(b)<b$.



Define $h(x)=g(x)-x$. It follows that
h is cont on $[a,b]$ and

$$h(a)=g(a)-a>0$$

$$h(b)=g(b)-b<0$$

By the Intermediate Value Thm, $\exists\, p\in(a,b) \ni h(p)=0$

Thus, $h(p)=0=g(p)-p \Rightarrow g(p)=p$ (Fixed point exists!)

Next, suppose $|g'(x)|\le k<1$ holds and p & q are both fixed pts in $[a,b]$ where $p\neq q$.

By MVT, $\exists\, \xi$ between p & q (and thus in $[a,b]$) with

$$\frac{g(p)-g(q)}{p-q}=g'(\xi) \Rightarrow |p-q|=|g(p)-g(q)|=|g'(\xi)|\,|p-q|\le k|p-q|<|p-q|$$

so $|p-q|<|p-q| \Rightarrow$ impossible! It must be $p=q$.

Examples

$g(x) = \ln\left(\frac{7}{x}\right)$ on $[1,2]$. Using EVT, $g'(x) = \frac{-1}{x} \neq 0$ on $[1,2]$

It follows that g is 1-1 on $[1,2]$ and will have max values at the endpts:

$$g(1) = \ln 7 \approx 1.946 \qquad \Rightarrow \quad g(x) \in [1.253, 1.946] \subset [1,2]$$
$$g(2) = \ln \tfrac{7}{2} \approx 1.253$$

Thus, a fixed pt exists in $[1,2]$

Further, $g'(x)$ is 1-1 function on $[1,2]$ and obtains max on endpts. Thus, $g'(1) = -1$, $g'(2) = -\frac{1}{2} \Rightarrow |g'(x)| \leq 1$

$K = 1 \Rightarrow$ This seems to fail But we can move the left pt of the interval. So choose $[1.5, 2]$ instead. $\Rightarrow$ It still satisfies the first part, and now $|g'(1.5)| \leq \frac{2}{3} = K < 1$

It is also possible to have a fixed pt, but not have the conditions of thm 2.1 satisfied. See book.

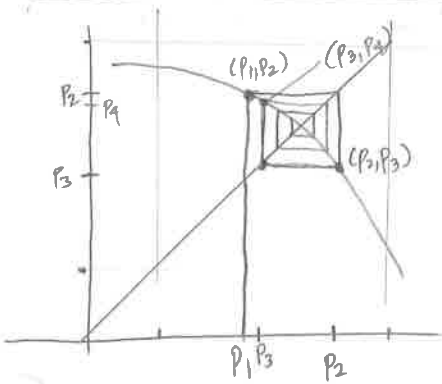
To approx a fixed pt, we choose an initial approx p_0 and then generate a sequence $\{p_n\}_{n=0}^{\infty}$ by letting

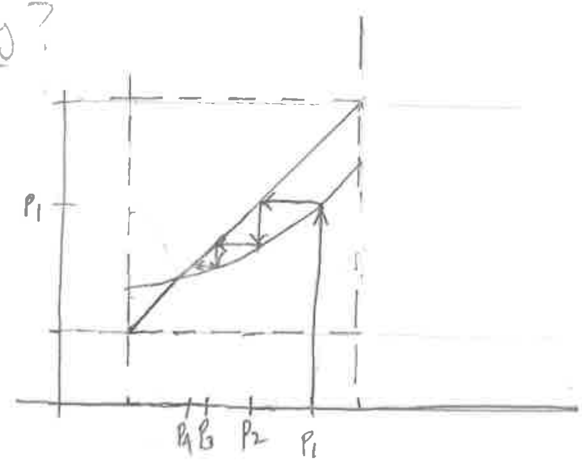
$$p_n = g(p_{n-1}) \quad \text{for each } n \geq 1$$

If the sequence converges and g is cont, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

How does it work visually?





How do you choose $g(p)$ for $f(x)=0$?

Suppose we want to solve $x^3+4x^2-10=0$

It has a root in $[1,2]$ (by IVT)

We could do:

$$x^3+4x^2-10=0 \Rightarrow 4x^2 = 10-x^3$$

$$x = \frac{\sqrt{10-x^3}}{2} \quad \text{(choose positive)}$$

let's call this one $g_3(x) = \frac{\sqrt{10-x^3}}{2}$

Here's some others!

$$x^3+4x^2-10=0 \Rightarrow g(x) = x-f(x)$$

$$g_1(x) = x-x^3-4x^2+10$$

$$x^3 + 4x^2 - 10 = 0$$

$$x^3 = 10 - 4x^2$$

$$x^2 = \frac{10 - 4x^2}{x} \Rightarrow x = \sqrt{\frac{10-4x^2}{x}} = g_2(x) \quad \text{(choose pos)}$$

$$x^3 + 4x^2 - 10 = 0$$

$$x^2(x+4) = 10$$

$$x^2 = \frac{10}{x+4}$$

$$x = \sqrt{\frac{10}{x+4}} = g_4(x)$$

$$g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$$

$$= \frac{2x^3+4x^2+10}{3x^2+8x}$$

Results:

$g_1(x) = x - x^3 - 4x^2 + 10 \Rightarrow$ fails to converge

$g_2(x) = \sqrt{\frac{10 - 4x^2}{x}} \Rightarrow$ fails to converge

$g_3(x) = \frac{\sqrt{10 - x^3}}{2} \Rightarrow$ converges in 30 iterations

$g_4(x) = \sqrt{\frac{10}{x+4}} \Rightarrow$ converges in 15 iterations

$g_5(x) = \frac{2x^3 + 4x^2 - 10}{3x^2 + 8x} \Rightarrow$ converges in 4 iterations!

Note that it is difficult to tell which converge by sight. How can we determine which will converge and how rapidly?

$\boxed{\text{Thm 2.3}}$ (Fixed pt thm)

Let $g \in C[a,b]$

$g(x) \in [a,b] \ \forall x \in [a,b]$

Suppose $g'(x)$ exists on (a,b) and

$|g'(x)| \leq K < 1$ for all $x \in (a,b)$

If p_0 is any number in $[a,b]$, then $\{p_n\}_{n=0}^{\infty}$ converges to the unique fixed pt in $[a,b]$ $\quad p_n = g(p_{n-1})$

proof: By Thm 2.2, a unique fixed pt exists. $p_n \in [a,b]$

By the MVT, we have

$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)||p_{n-1} - p| < K|p_{n-1} - p|$, where $\xi \in (a,b)$

So $|p_n - p| \leq K|p_{n-1} - p| \leq K^2|p_{n-2} - p| \leq \cdots \leq K^n|p_0 - p|$

Since $K < 1$, then $\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} K^n |p_0 - p| = 0 \Rightarrow p_n \to p$ as $n \to \infty$

Corollary **Bound for the error**

If g satisfies the hypothesis for Thm 2.3, then the bound for the error involve using p_n to approx p are given by:

$$|p_n - p| \le k^n \max\{p_0 - a,\ b - p_0\}$$

and

$$|p_n - p| \le \frac{k^n}{1-k} |p_0 - p_1|$$

proof: last inequality from previous proof:

$$|p_n - p| \le k^n |p_0 - p| \longrightarrow$$

$$\le k^n \max\{p_0 - a,\ b - p_0\}$$

Using the MVT, it follows that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le k |p_n - p_{n-1}| < \cdots < k^n |p_1 - p_0|$$

Thus, for $1 \le n < m$

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} - p_{m-2} + \cdots - p_{n+1} + p_{n+1} - p_n|$$

$$\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+2} - p_{n+1}| + |p_{n+1} - p_n|$$

$$\le K^{m-1}|p_1 - p_0| + K^{m-2}|p_1 - p_0| + \cdots + K^{n+1}|p_1 - p_0| + K^n|p_1 - p_0|$$

$$= K^n|p_1 - p_0| \underbrace{(1 + K + K^2 + \cdots + K^{m-n-1})}_{\text{geometric sum}}$$

So, let $m \to \infty$

$$\lim_{n\to\infty} |p_m - p_n| = |p - p_n| \le \lim_{m\to\infty} K^n |p_1 - p_0| \underbrace{\sum_{i=0}^{\infty} K^i}_{\frac{1}{1-K}} = \frac{K^n}{1-K}|p_1 - p_0|$$

Thus, the smaller the k, the faster the convergence!

What about the choices for g_1, g_2, g_3, g_4, g_5?

$g_1(x) = x - x^3 - 4x^2 + 10$

$g_1'(x) = 1 - 3x^2 - 8x$

No interval containing p for which $g'(x) < 1$

<u>move on</u>

$g_2(x) = \left(\frac{10-4x^2}{x}\right)^{1/2}$

<u>Bad</u> why?

doesn't map $[1,2]$ onto $[1,2]$

also $g_2'(p) = 3.4 > 1$ (k too big)

$g_3(x) \Rightarrow$ Note $[1,2]$ fails

but $[1,1.5]$ works

$g_3([1,1.5]) \subset [1,1.5]$

$g_3'(x) \leq .666$

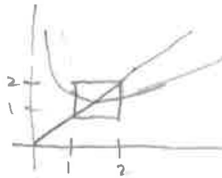
$g_4(x) = \sqrt{\frac{10}{4+x}}$

$g_4'(x) = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| \leq \frac{5}{\sqrt{10}\, 5^{3/2}} = \frac{\sqrt{2}}{10} \approx .14$

should converge fast!

What about g_5?

$g(1) = 1.45, \quad g(2) = 1.5$



$$g'(x) = \frac{2(3x-4)(x^3+4x^2-10)}{x^2(3x+8)^2} \Rightarrow \text{roots are } 1.365, 4/3$$

$$\left.\begin{aligned} g'(1) &= \frac{-70}{121} \\ g'(4/3) &= \frac{-7}{216} \\ g'(1.365) &= 0 \\ g'(2) &= \frac{5}{14} \end{aligned}\right\} |\max| = .5785 = K$$

Use $p_0 = 1$, and $p_1 = \frac{16}{11} = 1.45\overline{45}$

By thm 2.3 corollary,

$$|p_n - p| \le \frac{K^n}{1-K}|p_1 - p_0| = \frac{(\overset{70/121}{.5785})^n}{1-.5785}\underbrace{|1.45\overline{45} - 1|}_{\frac{5}{11}} = \frac{55}{51}\left(\frac{70}{121}\right)^n$$

To be guaranteed 10 digits, $\frac{55}{51}\left(\frac{70}{121}\right)^n = 10^{-8}$ $\quad n = 33.7$

$= 34.$

This still converges faster! Only 4 are needed.

Note that it is not really that close to .57 very long. Close to $1.33 = \frac{4}{3}$, the value is $\frac{7}{216}$. Using this gives: $p_0 = 4/3$, $K = \frac{7}{216}$, $p_1 = \frac{295}{216}$

$$\frac{K^n}{1-K}|p_1 - p_0| < tol \implies K^n < \frac{(1-K)\,tol}{|p_1 - p_0|}$$

$$n > \frac{\log\left[\frac{(1-K)\,tol}{|p_1 - p_0|}\right]}{\log K} = \underline{4.38105}$$

much better estimate!