$\boxed{2.4}$ Error Analysis for Iterative Methods

<u>DEF</u> Suppose $\{p_n\}_{n=0}^{\infty}$ is a seq. that converges to p.

If λ & α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda > 0 \text{, then } \{p_n\} \text{ is}$$

said to converge to p with order α, with asymptotic error const λ.

In general, a sequence of higher order converges to the solution faster than a seq. of lower order

1. $\alpha = 1 \Rightarrow$ linear convergence
2. $\alpha = 2 \Rightarrow$ quadratic "

EX:

Compare linear to Quadratic

Suppose $p_n \to 0$ (linear)

$\tilde{p}_n \to 0$ (Quadratic)

Suppose $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = .5 = \lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2}$

For sufficiently large n,

$$\frac{|p_{n+1}|}{|p_n|} \approx .5 \approx \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2}$$

For the linear convergent one,

$$|p_n| \approx 0.5|p_{n-1}| \approx 0.5(.5|p_{n-2}|) \approx .5(.5(.5|p_{n-3}|)) \approx \cdots \approx (.5)^n|p_0|$$

and for the quadratic conv. one,

$$|\tilde{p}_n| \approx 0.5|\tilde{p}_{n-1}|^2 \approx 0.5\left[0.5|\tilde{p}_{n-2}|^2\right]^2 \approx 0.5\left[0.5\left[0.5|\hat{p}_{n-3}|^2\right]^2\right]^2$$

$$\approx 0.5^3|\tilde{p}_{n-2}|^4 \approx 0.5^7|\hat{p}_{n-3}|^8 \approx (0.5)^{2^n-1}|\tilde{p}_0|^{2^n}$$

Note the table for the first 7 terms

n	p_n (Linear)	$\tilde{p}_n$ (quadratic)
1	5×10^{-1}	5×10^{-1}
2	2.5×10^{-1}	1.25×10^{-1}
3	1.25×10^{-1}	7.8125×10^{-3}
4	6.25×10^{-2}	3.0518×10^{-5}
5	3.125×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

Thm Let $g \in C[a,b]$ and suppose $g(x) \in [a,b]$ $\forall x \in [a,b]$

suppose g' is cont on (a,b) and $|g'(x)| \le k < 1$ $\forall x \in (a,b)$

If $g'(p) \neq 0$, then for any p_0 in $[a,b]$, the seq.

$$p_n = g(p_{n-1}),\ n \geq 1$$

converges only linearly to the unique fixed pt p. in $[a,b]$

<u>proof</u>:

we know from the fixed pt thm, that the seq conv. to p.

since g' exists on $[a,b]$, then, by the MVT

$$p_{n+1} - p = g(p_n) - g(p) = \underbrace{g'(\xi_n)(p_n - p)}_{\text{MVT}}$$

p is a fixed pt and

where ξ_n is between p_n & p. Since $p_n \to p$, then $\xi_n \to p$

Since g' is cont on $[a,b]$, we have

$$\lim_{n\to\infty} g'(\xi_n) = g'(p)$$

Thus,

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n\to\infty} g'(\xi_n) = g'(p) \quad \text{and}$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$$

Hence, if $g'(x) \neq 0$, then p_n converges linearly.

Thm 2.8

Let p be a solution to $x=g(x)$. Suppose $g'(p)=0$ and g'' is cont. & strictly bounded by M on an open interval I containing p. Then there exists a $\delta>0$ such that, for $p_0 \in [p-\delta, p+\delta]$, the seq. defined by $p_n = g(p_{n-1})$ converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1}-p| < \frac{M}{2}|p_n-p|^2$$

proof: Choose $\delta>0$ such that on the int $[p-\delta, p+\delta]$, contained in I, $|g'(x)| \le K < 1$, and g'' is continuous. Since $|g'(x)| \le K < 1$, it follows the terms of $\{p_n\}$ are contained in $[p-\delta, p+\delta]$. Expanding $g(x)$ in a Taylor poly.

about $x=p$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(\xi)}{2}(x-p)^2, \text{ where } \xi \text{ is between } x \,\&\, p.$$

Since $g(p)=p$ and $g'(p)=0$, then

$$g(x) = p + \frac{g''(\xi)}{2}(x-p)^2$$

when $x = p_n$

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n-p)^2, \text{ where } \xi_n \text{ is between } p_n \,\&\, p.$$

Since $|g'(x)| \le K < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ onto itself, it follows that p_n converges to p (by the fixed pt thm)

Since ξ_n is between $p_n \,\&\, p$, and $p_n \to p$, then $\xi_n \to p$ also

and $\displaystyle\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{|g''(p)|}{2}$ $\Rightarrow$ this implies quad convergence.

Since g'' is cont & bounded by M, then $|p_{n+1}-p| < \frac{M}{2}|p_n-p|^2$

Thm 2.7 & 2.8 say to choose a $g(x)$ so that $g'(p)=0$

deriv is zero at the fixed pt.

suppose $g(x) = x - \phi(x) f(x)$. Let's find what $\phi(x)$ needs to be to make $g'(p) = 0$

$$g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x)$$

$$g'(p) = 1 - \phi'(p) \underbrace{f(p)}_{=0} - \phi(p) f'(p)$$

$\downarrow$ force

$$0 = 1 - \phi(p) f'(p) \Rightarrow \phi(p) = \frac{1}{f'(p)}$$

So

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Since p is usually unknown, then let $p = p_{n-1}$ and

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{(Newton's method!)}$$

Start with converge to zero of $f(x) = x^2 - 2x + 1$ example.

DEF: A solution p of $f(x) = 0$ is said to be a zero of multiplicity m if f can be written as $f(x) = (x-p)^m g(x)$, for $x \neq p$, where $\lim_{x \to p} g(x) \neq 0$.

Thm 2.10 $f \in C'[a,b]$ has a simple zero at p in (a,b) iff $f(p) = 0$, but $f'(p) \neq 0$.

Thm 2.11 The function $f \in C^m[a,b]$ has a zero of mult m at p iff

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \text{ but } f^{(m)}(p) \neq 0.$$

$= x^2$ has a zero of mult 2 at 0

Ex: $f(x) = 2\cos x - 2 -$

Note: $f'(x) = -2\sin x - 2x$

$f''(x) = -2\cos x - 2$

$f(0) = 2\cos 0 - 2 - 0 = 2 - 2 = 0$

$f'(0) = -2\sin 0 - 2(0) = 0 - 0 = 0$

$f''(0) = -2(1) - 2 = -4 \neq 0.$

Modified Newton's Method

Newton's method does not converge quadratically when p is a zero of mult 2 or larger. in This case, we develop a modified function. Suppose f has a zero of mult. m at p. So $f(x) = (x-p)^m g(x)$

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m g(x)}{m(x-p)^{m-1} g(x) + (x-p)^m g'(x)}$$

$$= \frac{(x-p)\,g(x)}{m\,g(x) + (x-p)g'(x)}$$ $\Rightarrow$ this has a root at p, but it is a simple root.

So apply $f(x) = 0$ to $\mu(x) = 0$

Let $g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)/f'(x)}{f'(x)f'(x) - f(x)f''(x)}$

$[f'(x)]^2$

$$= x - \frac{f(x)}{f'(x)} \cdot \frac{[f'(x)]^2}{[f'(x)]^2 - f(x)\, f''(x)]}$$

$$= x - \frac{f(x)\, f'(x)}{(f'(x)]^2 - f(x) f'(x)}.$$

So
$$\boxed{x_{n+1} = x_n - \frac{f(x)\, f'(x)}{[f'(x)]^2 - f(x) f''(x)}}$$

Modified newton's method.