$\boxed{2.5}$ Accelerating Convergence

Two Methods $\Rightarrow$ Aitken's Δ^2 method
Steffensen's Method

We'd like to have better than linear convergence if possible. Here's a method (Aitken's Δ^2 method) that can accelerate the convergence.

So suppose $\{p_n\}$ is a linearly conv. sequence w/ $\lambda < 1$

We want to construct a new sequence $\{\hat{p}_n\}$ that converges faster. Suppose that $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ all agree on sign. Then

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}$$

we want to solve for p.

$$(p_{n+1} - p)^2 = (p_n - p)(p_{n+2} - p)$$

$$p_{n+1}^2 - 2p_{n+1}\,p + \cancel{p^2} = p_n p_{n+2} - p_n p - p_{n+2} p + \cancel{p^2}$$ (Move p's to other side)

$$p_{n+2}\,p + p_n p - 2p_{n+1}\,p = p_n p_{n+2} - p_{n+1}^2$$

$$(p_{n+2} - 2p_{n+1} + p_n)\,p = p_{n+2}\,p_n - p_{n+1}^2$$

$$p \approx \frac{p_{n+2}\,p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

To make it easier to use, we're going to do an add zero trick

Add zero

$$p \approx \frac{(p_{n+2}\,p_n - 2p_{n+1}\,p_n + p_n^2) - (p_{n+1}^2 - 2p_{n+1}\,p_n + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n}$$

add zero

$$p \approx \frac{p_n(p_{n+2} - 2p_{n+1} + p_n)}{p_{n+2} - 2p_{n+1} + p_n} - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

$$p \approx p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Let $\hat{p}_n = p$, then the sequence defined by $\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$ converges more rapidly than does the original sequence $\{p_n\}$

Example: To do AIKENS

In HP50, place sequence values in a list, then do:

```
3:  list        DELT  << ERR0 -> P0 P1 P2        (clear last error)
2:    3                  P2 2 P1 * - P0 +
1:  <<DELT>>             << P0 P1 P0 - 2 ^
run DOSUBS               IFERR / THEN + END - >> >>

                AIKENS  (Nothing else on stack!!
                  << DEPTH ->LIST 3 <<DELT>> DOSUBS >>
```

Use $G = \sqrt{\frac{10}{x+4}}$

n	p_n	$\tilde{p}_n$
1	1.41421356237	1.36523476888
2	1.35904021743	1.36523009045
3	1.36601821953	1.36523001466
4	1.36512974147	1.36523001343
5	1.3652427713	1.36523001342
6	1.36522839026	
7	1.36523021993	

DEF: Given a seq. $\Delta p_n = p_{n+1} - p_n$

the forward difference Δp_n is

$$\Delta p_n = p_{n+1} - p_n \quad , \; n \geq 0.$$

Higher powers are defined recursively:

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2.$$

It follows that

$$\Delta^2 p_n = \Delta(\Delta p_n) = \Delta(p_{n+1} - p_n) = \Delta p_{n+1} - \Delta p_n = p_{n+1} - p_n - p_n + p_{n+1}$$

Note that this means that Aitken's can be written as

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

Nice & concise!

Aitken's is more rapid in that

$\boxed{\text{Thm } 2.13}$

$$\lim_{n \to \infty} \frac{\overset{\text{Aitken's}}{\hat{p}_n} - p}{\underset{\text{original}}{p_n} - p} = 0$$

By applying a modified Aitken's method, we can accelerate linear to quadratic! This is called Steffenson's Method.

Here's how it works:

1. Apply fixed point method twice to a starting point
 Aitken's method to those three points.

2. Apply Aitken's
This point is now our "new starting" point,
and we go to step 1. Repeat until convergence

Here's an illustration between the two methods!

Aitken's Δ^2

fixed pt or some other method

new faster converging sequence

p_0

p_1 $\qquad \tilde{p}_0$

p_2 $\qquad \tilde{p}_1$

p_3 $\qquad \tilde{p}_2$

p_4 $\qquad \tilde{p}_3$

p_5 $\qquad \tilde{p}_4$

p_6

$\vdots$

Steffensen's Method

$\tilde{p}_0 \qquad p_0^{(0)}$

$p_1^{(0)} = g(p_0^{(0)})$

$p_2^{(0)} = g(p_1^{(0)})$

$\tilde{p}_1 \qquad p_0^{(1)} = \Delta^2(p_0^{(0)}, p_1^{(0)}, p_2^{(0)})$

$p_1^{(1)} = g(p_0^{(1)})$

$p_2^{(1)} = g(p_1^{(1)})$

$\tilde{p}_2 \qquad p_0^{(2)} = \Delta^2(p_0^{(1)}, p_1^{(1)}, p_2^{(1)})$

$\vdots$

Every third term is generated using Aitken's

$$f(x) = x^3 + 4x^2 - 10 = 0.$$

Do it using $\quad g(x) = \left(\frac{10}{x+4}\right)^{1/2}$

Do using calc. (or R)

```
S1X
  << DUP G DUP G DELT >>
DELT => given before
STEF
  << 0 'c' STO  0 SWAP
     DO SWAP DROP  DUP DUP G DUP G DELT
                           ST
     'c' 1 STO+  "1: ---" OVER + 9 DISP
    UNTIL  DUP2 - ABS  TOL ≤  C CLIM > OR  END DROP
  >>
```