$\boxed{3.2}$ Divided Differences

Previous section (Neville's / Lagrange) focused on interpolation at one point.

Divided difference methods generate the polynomials themselves.

$\boxed{\text{Def:}}$ nth Lagrange Polynomial in divided diff form

$$\boxed{\begin{aligned} P_n(x) = {} & a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots \\ & + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\ & \text{where the constants } a_0 \text{ are solved for.} \end{aligned}}$$

we know $P_n(x_k) = f(x_k)$ for $k = 0, \dots, k-1$

Let's find a_0. Plug x_0 into $P_n(x)$ (Note $P_n(x_0) = f(x_0)$ agree at this point)

$$P_n(x_0) = \boxed{a_0 = f(x_0)}$$

Let's find a_1. Plug x_1 into $P_n(x)$ (Remember $P_n(x_1) = f(x_1)$)

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1)$$

$$f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$a_1(x_1 - x_0) = f(x_1) - f(x_0)$$

$$\boxed{a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

Let's find a_2. Plug x_2 into $P_n(x)$ [Remember $P_n(x_2) = f(x_2)$]

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

At this point, it becomes more difficult (but possible) to do the algebra involved. However, we introduce a new notation:

$$f[x_i] = f(x_i), \quad f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

In particular, if we know the divided diff

$f[x_i, x_{i+1}, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}]$, then

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

With this, we can solve for each of the a_k's easier.

So, for example, back to the example

$$P_2(x) = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

same => see def of a_1 & $f[x_0, x_1]$

$$f(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

notation

$$f[x_2] - f[x_0] - f[x_0, x_1](x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

$$f[x_2] - f[x_1] + f[x_1] - f[x_0] - f[x_0, x_1](x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

trick

$$\underbrace{f[x_2]-f[x_1]}_{f[x_1,x_2](x_2-x_1)} + \underbrace{f[x_1]-f[x_0]}_{f[x_0,x_1](x_1-x_0)}$$

$$f[x_1,x_2](x_2-x_1) + \underbrace{f[x_0,x_1](x_1-x_0) - f[x_0,x_1](x_2-x_0)}_{x_0\text{'s cancel.}} = a_2(x_2-x_0)(x_2-x_1)$$

$$f[x_1,x_2](x_2-x_1) + f[x_0,x_1](x_1-x_2) = a_2(x_2-x_0)(x_2-x_1)$$

So

$$a_2 = \frac{f[x_1,x_2](x_2-x_1) + f[x_0,x_1]\underbrace{(x_1-x_2)}_{=(-(x_2-x_1))}}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2-x_0} = f[x_0,x_1,x_2]$$

In general,

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x-x_0)\cdots(x-x_{k-1})$$

Divided Difference Table

x	$f(x)$	First Divided Differences	Second DD
x_0	$f[x_0]$		
		$f[x_0,x_1] = \dfrac{f[x_1]-f[x_0]}{x_1-x_0}$	
x_1	$f[x_1]$		$f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$
		$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{\ldots}$	$\dfrac{f[x_2,x_3]-f[x_1,x_2]}{\ldots}$

$$x_2 \qquad f[x_2] \qquad f[x_1,x_2] = \frac{\cdots}{x_2 - x_1} \qquad f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3 - x_1}$$

$$f[x_2,x_3] = \frac{f[x_3]-f[x_2]}{x_3 - x_2}$$

$$x_3 \qquad f[x_3] \qquad f[x_2,x_3,x_4] = \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4 - x_2}$$

$$f[x_3,x_4] = \frac{f[x_4]-f[x_3]}{x_4 - x_3}$$

$$x_4 \qquad f[x_4] \qquad f[x_3,x_4,x_5] = \frac{f[x_4,x_5]-f[x_3,x_4]}{x_5 - x_3}$$

$$f[x_4,x_5] = \frac{f[x_5]-f[x_4]}{x_5 - x_4}$$

$$x_5 \qquad f[x_5]$$

you can also define

3 Third DD

2 Fourth DD

& 1 Fifth DD

Algorithm 3.2 Newton's Interpolary Divided - Diff Formula

Note that $P(x) = \sum_{i=0}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j)$ ⇒ we're after the numbers on the diagonal. ($F_{i,0} = f(x_i)$)

outputs the polynomial.

Thm 3.6

Suppose $f \in C^n[a,b]$, x_i's are distinct $\in [a,b]$. $\exists\ \xi \ni$

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Now, reformulate

$$(*) \quad P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k](x-x_0)\cdots(x-x_{k-1})$$

with equal spacing let $h = x_{i+1} - x_i$ (common for all)

Let $x = x_0 + sh$ (point to interpolate)

$x_i = x_0 + ih$ and

$x - x_i = (s-i)h$

Then $(*)$ becomes

$$P_n(x) = P_n(x_0+sh) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

$$+ \cdots + f[x_0,x_1,\cdots,x_k](x-x_0)(x-x_1)\cdots(x-x_{k-1}) + \cdots + \text{last one.}$$

Note: $(x-x_0)(x-x_1)\cdots(x-x_{k-1}) = (sh)(s-1)h\,(s-2)h\cdots(s-k+1)h$

$$= s(s-1)\cdots(s-k+1)\,h^k$$

To simplify notation, define an extension of the binomial coefficient as

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}, \quad \text{where } s \in \mathbb{R} \text{ (any real \#)}$$

Thus,

$$(x-x_0)\cdots(x-x_{k-1}) = \binom{s}{k} k!\, h^k \quad \text{and}$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} f[x_0,\dots,x_k]\binom{s}{k} k!\, h^k}$$

Newton's Forward divided diff formula

If we use Aiken's Δ^2 operator, we can make a simplification to the notation for the formula. First, note

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{\Delta f(x_0)}{h} \quad \text{and}$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}}{2h} = \frac{\Delta^2 f(x_0)}{2h^2}$$

In general

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!\,h^k} \Delta^k f(x_0), \quad \text{so NFDDM}$$

becomes

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] \binom{s}{k} k!\,h^k$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f(x_0)}$$

Newton Forward-Difference formula.

You can also reorder the indices from the back to the front. $x_n, x_{n-1}, \ldots, x_0$. In this case, we get

... backward divided difference formula

Newton backward divided difference form

$$P_n(x) = f[x_n] + f[x_{n-1}, x_n](x-x_n) + f[x_{n-2}, x_{n-1}, x_n](x-x_n)(x-x_{n-1})$$
$$+ \cdots + f[x_0, \ldots, x_n](x-x_n)(x-x_{n-1}) \cdots (x-x_1)$$

Using equal spacing (like before) yields

$$P_n(x) = f[x_n] + sh\, f[x_{n-1}, x_n] + s(s+1)h^2 f[x_{n-2}, x_{n-1}, x_n] + \cdots + s(s+1)\cdots(s+n-1)h^n f[x_0, \ldots, x_n]$$

DEF: Backwards Diff operator $\nabla P_n \equiv P_n - P_{n-1}$ and

$$\nabla^k P_n = \nabla(\nabla^{k-1} P_n)$$

This makes $f[x_{n-1}, x_n] = \frac{1}{n} \nabla f(x_n)$, $f[x_{n-2}, x_{n-1}, x_n] = \frac{1}{2h^2} \nabla^2 f(x_n)$

and, in general, $f[x_{n-k}, \ldots, x_n] = \frac{1}{k!\,h^k} \nabla^k f(x_n)$.

Which means

$$P_n(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n) + \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!}\nabla^n f(x_n)$$

Note that the binomial coefficient idea doesn't seem to work because the multiply is going up (not down). But it can still be done using the same extension!

So

$$\binom{-s}{k} = \frac{-s(-s-1)(-s-2)\cdots(-s-k+1)}{k!} = \frac{(-1)^k s(s+1)(s+2)\cdots(s+k-1)}{k!}$$

Thus, Newton's Backward Difference formula is

$$P_n(x) = \sum^{n} (-1)^k \binom{-s}{\ } \nabla^k f(x$$

$$P_n(x) = \sum_{k=0} (\;) \binom{\;}{k} \nabla f(x_n)$$

Examples

Choose Normal table from 1 to 2 separated by .2