3.3 Hermite Polynomials

Osculating Polynomials: generalisation of both Taylor Polynomials & Lagrange Polynomials

What are osculating polynomials?

DEF: Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct numbers in $[a,b]$ and m_i be a non-negative integer associated with x_i for $i=0,1,\ldots,n$. Let

$$m = \max_{0 \le i \le n} m_i \quad \text{and} \quad f \in C^m[a,b]$$

The osculating polynomial approximating f is the polynomial P of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}, \quad \text{for each } i=0,1,\ldots,n \text{ and } k=0,1,\ldots,K$$

Note: $n=0 \Rightarrow$ osculating polynomial $= m_0^{\text{th}}$ Taylor polynomial

$m_i = 0 \ \forall i \Rightarrow$ osculating polynomial $= n^{\text{th}}$ Lagrange poly on $x_0, \dots, x_n$

When $m_i = 1$ for each $i = 0, 1, \dots, n$, then we have what we call the Hermite Polynomials. They not only agree with the function $f(x)$ at the pts $x_0, x_1, \dots, x_n$, but they also agree with $f'(x)$ at the same points. The Hermite polynomial has the same "shape" as $f(x)$ in the sense that the tangent lines to both are the same.

We restrict our study of osculating poly. to Hermite polys.

Thm If $f \in C'[a,b]$ and $x_0, \dots, x_n \in [a,b]$ are distinct, the unique polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x), \text{ where}$$

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}] L^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$$

In this context, $L_{n,j}$ denotes the j^{th} Lagrange coefficient polynomial of degree n defined in Eq 3.3

Moreover, if $f \in C^{(2n+2)}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \underbrace{\frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)}_{R_{2n+1}(x)}, \quad \text{for } \xi \in (a,b)$$

$R_{2n+1}(x)$ (remainder term)

<u>Proof</u>: Shows how the Hermite polynomial agrees with $f(x)$ at the nodes and $f'(x)$ agrees with $P'(x)$ at the nodes

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<u>Example</u>: Use the following table to construct the Hermite Polynomial for the data $f(x) = \sin x$

Hermite Polynomial for the data: $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

K	x_K	$f(x_K)$	$f'(x_K)$
0	0.0	1	0
1	0.1	.99833417	-.03330001
2	0.2	.99334665	-.06640038

First, compute the Lagrange polynomials: (and derivs)

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-.1)(x-.2)}{(0-.1)(0-.2)} = 50x^2-15x+1 \qquad L'_{2,0}(x) = 100x-15$$

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-.2)}{(.1-0)(.1-.2)} = -100x^2+20x \qquad L'_{2,1}(x) = -200x+20$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-.1)}{(.2-0)(.2-.1)} = 50x^2-5x \qquad L'_{2,2}(x) = 100x-5$$

Next, the polynomials $H_{2,j}$ & $\hat{H}_{2,j}$

$$H_{2,0} = \left[1 - 2(x - x_0)L'_{2,0}(\underbrace{x_0}_{0})\right] L^2_{2,0}(x)$$

$$= \left[1 - 2(x - 0)(-15)\right](50x^2 - 15x + 1)^2$$

$$\boxed{H_{2,0} = (1 + 30x)(50x^2 - 15x + 1)^2}$$

$$\hat{H}_{2,0} = (x - x_0)L^2_{2,0}(x)$$

$$\boxed{\hat{H}_{2,0} = x\,(50x^2 - 15x + 1)^2}$$

$$H_{2,1} = \left[1 - 2(x - x_1)L'_{2,1}(x_1)\right]L^2_{2,1}(x)$$

$$= \left[1 - 2\left(x - \tfrac{1}{10}\right)\underbrace{\left(-200\left(\tfrac{1}{10}\right) + 20\right)}_{0}\right](-100x^2 + 20x)^2$$

$$\boxed{H_{2,1} = (-100x^2 + 20x)^2}$$

$$\hat{H}_{2,1} = (x - x_1)L^2_{2,1}(x)$$

$$= \left(x - \tfrac{1}{10}\right)(-100x^2 + 20x)^2$$

$$\boxed{\hat{H}_{2,1} = \tfrac{1}{10}(10x - 1)(-100x^2 + 20x)^2}$$

$$H_{2,2} = \left[1 - 2(x - x_2)L'_{2,2}(x_2)\right]L^2_{2,2}(x)$$

$$= \left[1 - 2\left(x - \tfrac{2}{10}\right)\left(100\left(\tfrac{2}{10}\right) - 5\right)\right](50x^2 - 5x)^2$$

$$\hat{H}_{2,2} = (x - x_2)(50x^2 - 5x)^2$$

$$= \left(x - \frac{1}{}\right)(50x^2 - 5x)^2$$

$$\boxed{H_{2,2} = -(30x-7)(50x^2-5x)^2} \qquad \boxed{\hat{H}_{2,2} = \frac{1}{5}(5x-1)(50x^2-5x)^2}$$

Finally $(n=2)$ and

$$H_5(x) = f(x_0)H_{2,0}(x) + f(x_1)H_{2,1}(x) + f(x_2)H_{2,2}(x)$$
$$+ f'(x_0)\hat{H}_{2,0}(x) + f'(x_1)\hat{H}_{2,1}(x) + f'(x_2)\hat{H}_{2,2}(x)$$

$$= (1)(1+30x)(50x^2-15x+1)^2 + .99833417(-100x^2+20x)^2 + .99334665(-(30x-7)(50x^2-5x)^2$$
$$+ 0\cdot x\cdot(50x^2-15x+1)^2 + -.0333001\left(\frac{1}{10}(10x-1)(-100x^2+20x)^2\right) + -.06640038\left(\frac{1}{5}(5x-1)(50x^2-5x)^2\right)$$

$=$

Note:

$H_5(.05) = .99958338\underline{7031}$ (real answer $.99958338\underline{5414}$

Although we did this pretty clearly, it wasn't all that easy. If you increase n, then it becomes even more unwieldy.

We can modify the Newton's interpolary divided difference formula to do this. Here's the modification

z	$f(z)$	first DD	second DD
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		same
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		same
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_2] = f(x_1)$		same
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		same

$z_5 = x_2 \qquad f[z_5] = f(x_2)\ ; \quad f[z_4, z_5] = f'(x_2)$

So, for our example,

	z	$f(z)$	first DD	2nd DD	3rd DD	4th DD	5th
0	0	1					
1	0	1	0				
2	.1	.99833417	−.0166583	−.166583			
3	.1	.99833417	−.03330001	−.1664171	.001659		
4	.2	.99334665	−.0498752	−.1657519	.003326	.008335	
5	.2	.99334665	−.06640038	−.1652518	.005001	.008375	.0002

So

$$H_5(.05) = 1 + (.05-0)(0) + (.05-0)^2(-.166583) + (.05-0)^2(.05-.1)(.001659)$$

$$+ (.05-0)^2(.05-.1)^2(.008335) + (.05-0)^2(.05-.1)^2(.05-.2)(.0002)$$

$$= .999583387031_{2} \quad \text{(same as before)}$$