$\boxed{4.2}$ Richardson's Extrapolation

Used to generate results of high accuracy by using low order formulas.

Extrapolation is applied when an error term has a predictable form, one that depends on a parameter, usually a stepsize h.
For example, suppose $N(h)$ is a formula that approximates an unknown value M and that $N(h)$ has $O(h)$ truncation error in the form

$$(*) \quad M = \underbrace{N(h)}_{\text{used to approx } M} + \underbrace{K_1 h + K_2 h^2 + K_3 h^3 + \cdots}_{\text{remainder}}, \text{ for unspecified constants } K_i.$$

$\Rightarrow$ assume $h > 0$ can be arbitrarily chosen, and $N(h)$ becomes more accurate as h becomes small.

$\Rightarrow$ objective: Improve $N(h)$ from order $O(h)$ to a formula of higher order ($O(h^2)$) or better.

First step: replace h by $\frac{h}{2}$ in (*) above. Then

$$(**) \quad M = N\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots$$

Combine (**) and (*) to eliminate K_1. (2(**) − (*))

$$2M - M = 2N\left(\frac{h}{2}\right) + K_1 h + K_2 \frac{h^2}{2} + K_3 \frac{h^3}{4} + \cdots$$
$$- N(h) - K_1 h - K_2 h^2 - K_3 h^3 - \cdots$$

$$M = \underbrace{2N\left(\frac{h}{2}\right) - N(h)}_{\text{call } N_2(h)} - \frac{K_2 h^2}{2} - \frac{3}{4} K_3 h^3$$

and call $N \equiv N_1$

Thus, $M = N_2(h) - \underbrace{\frac{1}{2} K_2 h^2 - \frac{3}{4} K_3 h^3}_{O(h^2)}$ (***)

Replace h by $\frac{h}{2}$ in (***). Then

$$M = N_2\left(\frac{h}{2}\right) - \frac{1}{2} K_2 \frac{h^2}{4} - \frac{3}{4} K_3 \frac{h^3}{8} + \cdots \quad (****)$$

Combine (***) and (****) to eliminate h^2 term.

So (4 (****) − (***))

$$4M - M = 4N_2\left(\frac{h}{2}\right) - N_2(h) \underbrace{- \frac{3}{8} K_3 h^3 + \frac{3}{4} K_3 h^3}_{\frac{3}{8} K_3 h^3}$$

$$3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) + \frac{3}{8} K_3 h^3$$

$$M = \underbrace{\frac{4}{3} N_2\left(\frac{h}{2}\right) - \frac{N_2(h)}{3}}_{N_3(h)} + \underbrace{\frac{1}{8} K_3 h^3}_{O(h^3)}$$

Note that we can continue this process!

$$N_4(h) = \frac{8}{7} N_3\left(\frac{h}{2}\right) - \frac{N_3(h)}{7}$$

In general, if $M = N(h) + \sum_{1}^{m-1} K_j h^j + O(h^m)$, then

$$\boxed{N_{j+1}(h) = \frac{2^j N_j\left(\frac{h}{2}\right) - N_j(h)}{2^j - 1} = N_j\left(\frac{h}{2}\right) + \frac{N_j\left(\frac{h}{2}\right) - N_j(h)}{2^j - 1}}$$

These approximations can easily be placed in a table (evaluated in the circled order)

$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$
$N_1(h)$ 1			
$N_1(\frac{h}{2})$ 2	$N_2(h)$ 3		
$N_1(\frac{h}{4})$ 4	$N_2(\frac{h}{2})$ 5	$N_3(h)$ 6	
$N_1(\frac{h}{8})$ 7	$N_2(\frac{h}{4})$ 8	$N_3(\frac{h}{2})$ 7	$N_4(h)$ (8)

<u>Example</u>: Use $f'(x_0) = \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{N_1(h)}$ and Richardson's Extrapolation

to improve upon the approx for

$f(x) = xe^x$ at $x_0 = 2$ real ans $f'(2) = 3e^2$

Let $h = .2$, then

$N_1(.2) = 25.384587504$

$N_1(.1) = 23.708446185$ $\quad N_2(.2) = 22.032304866$

$N_1(.05) = 22.9217014$ $\quad N_2(.1) = 22.1349566615$ $\quad N_3(.2) = 22.1691738647$

$N_1(.025) = 22.5404986$ $\quad N_2(.05) = 22.1592958$ $\quad N_3(.1) = 22.1674088617$

N_4

$$N_2(.2) = N_1(.1) + \frac{N_1(.1) - N_1(.2)}{2^{2-1} - 1} = 2N_1(.1) - N_1(.2)$$

$$= 2(23.\cdots) - 25.3\ldots$$

$$= 22.032304866$$

$$N_2(.1) = N_1(.05) + \frac{N_1(.05) - N_1(.1)}{2 - 1} = 22.1349566615$$

$$N_3(.2) = N_2(.1) + \frac{N_2(.1) - N_2(.2)}{3} = 22.1691738647$$

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We can use this process to derive better formulas. For example, expand $f(x)$ in a taylor series about $x = x_0$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + O(x-x_0)^4$$

evaluate at $x = x_0 + h$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + O(h^4)$$

Solving for $f'(x_0)$ yields

$$(*) \quad f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{f''(x_0)}{2}h - \frac{f^{(3)}(x_0)}{3!}h^2 + O(h^3)$$

Our goal is to eliminate the h term. So, replace h by $2h$ in $(*)$ gives

$$(**) \quad f'(x_0) = \frac{f(x_0+2h) - f(x_0)}{2h} - f''(x_0)h - \frac{4}{6}f^{(3)}(x_0)h^2 + O(h^3)$$

Multiply (*) by 2 and subtract (**)

$$2f'(x_0) - f'(x_0) = \frac{2f(x_0+h) - 2f(x_0)}{h} - \left(\frac{f(x_0+2h) - f(x_0)}{2h}\right) + \underbrace{\left(-\frac{2}{6} + \frac{4}{6}\right)}_{\frac{1}{3}} f^{(3)}(x_0)\,h^2 + o(h^3)$$

$$f'(x_0) = \frac{4f(x_0+h) - 4f(x_0)}{2h} + \frac{f(x_0) - f(x_0+2h)}{2h} + \frac{1}{3} f^{(3)}(x_0)\,h^2 + o(h^3)$$

$$= \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + \frac{1}{3} f^{(3)}(x_0)\,h^2 + o(h^3)$$

$$= \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{1}{3} f^{(3)}(x_0)\,h^2 + o(h^3)$$

Other formulas can be generated in a similar manner

This topic is used throughout the text!