$\boxed{4.4}$ Composite Numerical Integration

Newton-Cotes generally unsuitable for large intervals

- inaccurate over large int (oscillatory polynomials)
- coefficients are difficult to compute

Instead, apply these piecewise.

For example, consider the simple integral $\int_0^4 e^x\,dx = e^4 - 1$
$= 53.59815$

If we use Simpson's with $h=2$, then

$$\int_0^4 e^x\,dx \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] = \frac{2}{3}\left[e^0 + 4e^2 + e^4\right] = 56.96958$$

The Error is pretty large here $= 53.59815 - 56.96958 = -3.17143$

Let's divide $[0,4]$ into $[0,2]$ and $[2,4]$ and apply rule to each interval

$$\int_0^4 e^x\,dx \approx \frac{h_1}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] + \frac{h_2}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$

$$= \frac{1}{3}\left[e^0 + 4e^1 + e^2\right] + \frac{1}{3}\left[e^2 + 4e^3 + e^4\right]$$

$$= 53.86385 \Rightarrow \text{error} \quad 53.59815 - 53.86385 = -.26570$$

Can we continue? Yes! It can get lots better by increasing the number of points.

In general, $h = \frac{b-a}{n}$, $x_j = a + jh$ for $i = 0, \ldots, n$,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + 4f(x_3) + f(x_4)\right] + \cdots$$

$$+ \frac{h}{3}\left[f(x_{n-3}) + 4f(x_{n-2}) + f(x_{n-1})\right] + \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$+ \sum \text{error terms}$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\}$$

$$= \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

begins ↓ ends ↙

Note that comp Simp Rule is $1, 4, 2, 4, 2, 4, 1$

Analyze the error term:

$$E(f) = -\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \qquad \text{Note: } x_{2j-2} < \xi_j < x_{2j}$$

If $f \in C^4[a,b]$, then by the EVT, $f^{(4)}$ assumes it's max in $[a,b]$. We can create a bound for it:

$$\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x), \quad j=1,\dots,n/2$$

So

$$\frac{n}{2} \min_{x\in[a,b]} f^{(4)}(x) \le \sum_{1}^{n/2} f^{(4)}(\xi_j) \le \frac{n}{2} \max_{x\in[a,b]} f^{(4)}(x)$$

or

$$\min_{x\in[a,b]} f^{(4)}(x) \le \frac{2}{n} \sum_{1}^{n/2} f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x)$$

... $f^{(4)}$ is cont then $\exists\, \mu \in (a,b)$

Since we have this inequality, and f is cont,
Such that
$$f^{(4)}(\mu) = \frac{2}{n}\sum_{1}^{n/2} f^{(4)}(\xi_j) \qquad \text{(Intermediate Value thm)}$$

It follows that
$$E(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{h^5}{90}\left(\frac{n}{2}\right) f^{(4)}(\mu) = -\frac{nh^5}{180} f^{(4)}(\mu)$$

since $h = \frac{b-a}{n}$, then
$$E(f) = -\frac{\cancel{n} h^4 \frac{(b-a)}{\cancel{n}}}{180} f^{(4)}(\mu) = -\frac{(b-a)}{180} h^4 f^{(4)}(\mu)$$

Summarizing, Simpson's Rule is (n must be even)
$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\mu),$$

other Rules may also be set up:

Composite Trapezoid Rule

$(n \in \mathbb{Z}^+)$

$h = \frac{b-a}{n} \qquad x_j = a + jh \quad , \; j = 0, \cdots, n$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f^{(2)}(\mu)$$

Composite Midpt Rule

(n must be even)

$$\int_a^b f(x)\,dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6} h^2 f^{(2)}(\mu)$$

Note: Integration is stable unlike differentiation

As $h \to 0$, the round off error is bounded. This is good!