Romberg Integration

Combine Composite Trap & Richardson's Extrapolation

To help in this process, define Trap with

$h_k = \dfrac{b-a}{m_k}$, $m_k = 2^{k-1}$. Note $h_k = \dfrac{b-a}{2^{k-1}} = \dfrac{1}{2}\left(\dfrac{b-a}{2^{k-2}}\right) = \dfrac{1}{2} h_{k-1}$

step size   intervals

So $h_k = 2 h_{k+1}$ &

$h_k = \dfrac{h_{k-1}}{2}$

$$(*) \quad \int_a^b f(x)\,dx = \frac{h_k}{2}\left[f(a) + f(b) + 2\left(\sum_{i=1}^{2^{k-1}-1} f(\underbrace{a + i h_k}_{x_i})\right)\right] - \frac{b-a}{12} h_k^2 f''(\mu_k)$$

Denote $R_{k,1}$ as the part above used as Trap Rule (not error).

$$R_{1,1} = \frac{h_1}{2}\left[f(a) + f(b)\right]$$

$$R_{2,1} = \frac{h_2}{2}\left[f(a) + f(b) + 2\left(\sum_{1} f(a + i h_2)\right)\right]$$

$$= \frac{h_2}{2}\left[f(a) + f(b)\right] + \underbrace{h_2}_{\frac{h_1}{2}} f(a + h_2)$$

$$= \frac{1}{2}\left[\underbrace{h_2}_{\frac{h_1}{2}}\left[f(a) + f(b)\right] + h_1 f(a + h_2)\right]$$

$$= \frac{1}{2}\left[R_{1,1} + h_1 f(a + h_2)\right]$$

$$R_{3,1} = \frac{h_3}{2}\left[f(a) + f(b) + 2\left(\sum_{i=1}^{4} f(a + i h_3)\right)\right] \qquad \left(h_3 = \frac{h_2}{2}\right)$$

$$= \frac{h_2}{4}\left[f(a) + f(b) + 2f(a + h_3) + 2f(a + \underbrace{2h_3}_{h_2}) + 2f(a + 3h_3)\right]$$

$$= \underbrace{\frac{h_2}{4}}_{\frac{h_1}{8}}\left[f(a) + f(b) + 2f(a + h_2)\right] + \frac{h_2}{4}\left[2f(a + h_3) + 2f(a + 3h_3)\right]$$

$$= \frac{1}{2}\left[\frac{h_1}{4}\left(f(a)+f(b)+2f(a+h_2)\right) + h_2\left(f(a+h_3)+f(a+3h_3)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\left[\underbrace{\frac{h_1}{2}\left(f(a)+f(b)\right)}_{R_{1,1}} + h_1 f(a+h_2)\right] + h_2\left(f(a+h_3)+f(a+3h_3)\right)\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{2}\left[R_{1,1} + h_1 f(a+h_2)\right]}_{R_{2,1}} + h_2\left(f(a+h_3)+f(a+3h_3)\right)\right]$$

$$R_{3,1} = \frac{1}{2}\left[R_{2,1} + h_2 \sum_{i=1}^{2^{3-2}} f(a+(2i-1)h_3)\right]$$

In general,

$$\boxed{R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1} \sum_{i=1}^{2^{k-2}} f(a+(2i-1)h_k)\right]}$$

All we've done is define Trap Rule for 2^k subdivisions.

We can combine with Richardson's as it halves the step size on each step.

Note: It can be shown (very hard) that if $f \in C^{\infty}[a,b]$, then Comp Trap can be written as

$$(*) \qquad \int_a^b f(x)\,dx - R_{k,1} = \sum_{i=1}^{\infty} K_i h_k^{2i} = K_1 h_k^2 + K_2 h_k^4 + \cdots$$

The $k+1^{st}$ one is

$$(**) \qquad \int_a^b f(x)\,dx - R_{k+1,1} = \sum_{i=1}^{\infty} K_i h_{k+1}^{2i} = \frac{K_1 h_k^2}{4} + \frac{K_2 h_k^4}{16} + \cdots$$

$4(**) - (*)$ gives

$$4\int_a^b f(x)\,dx - 4R_{k+1,1} - \int_a^b f(x)\,dx + R_{k+1,1} = -\frac{3}{4} h_k^4 + O(h_k^6)$$

$$3\int_a^b f(x)\,dx = 4R_{k+1,1} - R_{k+1,1}$$

$$\int_a^b f(x)\,dx = \frac{3R_{k+1,1}}{3} + \frac{R_{k+1,1} - R_{k,1}}{3} = R_{k+1,1} + \frac{R_{k+1,1} - R_{k,1}}{3}$$

Define

$$R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{3}$$

In general,

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$$

We can generate a table now, where

$R_{1,1}$

$R_{2,1}$ $\quad$ $R_{2,2}$

$$
\begin{array}{ccccc}
R_{3,1} & R_{3,2} & R_{3,3} & & \\
\vdots & \vdots & & \ddots & \\
R_{n,1} & R_{n,2} & R_{n,3} & \cdots & R_{n,n}
\end{array}
$$

Algorithm