Gaussian Quadrature

Note that Newton-Cotes uses equally spaced points
why not have them be unequal? Maybe it can be
lots better?

We want to use the same formula, but pick x_i's as well!

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} c_i f(x_i) \Rightarrow \text{exact results for polynomials of the largest degree.}$$

Since we want to estimate $c_1, c_2, \ldots, c_n$ and
$x_1, x_2, \ldots, x_n$, then

We'd like to estimate $2n$ parameters, To fit a polynomial,
we need degree $2n-1$. We'll focus currently on $[-1,1]$ for interval

Example: Let $n=2$. Then we need to estimate

$c_1, c_2, x_1, x_2 \Rightarrow$ degree $2(n)-1 = 2(2)-1 = 3$.

So that $\int_{-1}^{1} f(x) = c_1 f(x_1) + c_2 f(x_2)$ is exact for polynomial of degree three or less. If we find what the constants are for $f(x)=1$, x, x^2, and x^3, then it will work for all polys of degree 3 or less

for $f(x)=1$ $\quad c_1 \cdot 1 + c_2 \cdot 1 = \int_{-1}^{1} 1\,dx = 1-(-1) = 2$

for $f(x)=x$ $\quad c_1 \cdot x_1 + c_2 \cdot x_2 = \int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{(-1)}{2} = 0$

for $f(x)=x^2$ $\quad c_1 \cdot x_1^2 + c_2 \cdot x_2^2 = \int_{-1}^{1} x^2\,dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$

for $f(x)=x^3 \Rightarrow c_1 x_1^3 + c_2 x_2^3 = 0$

Solving the system.

So we need to solve

① $c_1 + c_2 = 2$

② $c_1 x_1 + c_2 x_2 = 0$

③ $c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}$

④ $c_1 x_1^3 + c_2 x_2^3 = 0$

$\Longrightarrow$ ② = ④

$$c_1 x_1 + c_2 x_2 = c_1 x_1^3 + c_2 x_2^3$$

$$c_2 x_2 - c_2 x_2^3 = c_1 x_1^3 - c_1 x_1$$

$$c_2 x_2 (1 - x_2^2) = c_1 x_1 (x_1^2 - 1) \quad ⑤$$

$c_1 = 2 - c_2 \Rightarrow$ plug into ②

$$(2 - c_2) x_1 + c_2 x_2 = 0$$

$$2x_1 - c_2 x_1 + c_2 x_2 = 0$$

$$2x_1 = c_2 (x_1 - x_2)$$

$$x_1 - x_2 = \frac{2x_1}{c_2} \quad ⑥$$

or $c_2 = 2 - c_1$

$$c_1 x_1 + (2 - c_1) x_2 = 0$$

$$c_1 (x_1 - x_2) = -2x_2$$

$$x_1 - x_2 = \frac{-2x_2}{c_1}$$

$$\frac{2x_1}{c_2} = \frac{-2x_2}{c_1} \Rightarrow x_1 c_1 = -x_2 c_2 \Rightarrow \text{plug into } (5)$$

$$\Rightarrow \cancel{c_2}\cancel{x_2}(1-x_2^2) = -\cancel{c_2}\cancel{x_2}(x_1^2-1)$$

$$x_2^2 = x_1^2 \Rightarrow \boxed{x_1 = -x_2}$$

Thus, (6) says

$$\underbrace{x_1-(-x_1)}_{2x_1} = \frac{2x_1}{c_2} \Rightarrow \boxed{c_2 = 1} \Rightarrow \boxed{c_1 = 2 - c_2 = 2 - 1 = 1}$$

Last, (3) says

$$\underbrace{c_1}_{1}\underbrace{x_1^2}_{x_2^2} + \underbrace{c_2}_{1} x_2^2 = \frac{2}{3}$$

$$2x_2^2 = \frac{2}{3} \Rightarrow x_2^2 = \frac{1}{3} \Rightarrow \boxed{x_2 = \frac{1}{\sqrt{3}},\ x_1 = -\frac{1}{\sqrt{3}}}$$

Note $\displaystyle\int_{-1}^{1} f(x)\,dx = f\Big(\overset{-\frac{1}{\sqrt{3}}}{\frac{-\sqrt{3}}{3}}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)$ is exact for degree 3 or less. Try it!

$$\int_{-1}^{1} x^3 + 2x^2\,dx = \frac{x^4}{4} + \frac{2}{3}x^3\Big]_{-1}^{1} = \Big(\frac{1}{4} - \frac{1}{4}\Big) + \frac{2}{3} - \Big(-\frac{2}{3}\Big) = \frac{4}{3}$$

So $f\big(\frac{-\sqrt{3}}{3}\big) + f\big(\frac{\sqrt{3}}{3}\big) = \Big(\frac{-\sqrt{3}}{3}\Big)^3 + 2\Big(\frac{-\sqrt{3}}{3}\Big)^2 + \Big(\frac{\sqrt{3}}{3}\Big)^3 + 2\Big(\frac{-\sqrt{3}}{3}\Big)^2$

$$= \underbrace{-\Big(\frac{\sqrt{3}}{3}\Big)^3 + \Big(\frac{\sqrt{3}}{3}\Big)^3}_{0} + \frac{2}{3} + \frac{2}{3}$$

$$= \frac{4}{3} \quad \checkmark$$

This technique can be used for higher number of modes and coefficients, but it's more difficult than another method.

In section 8.2&8.3, we learn more about orthogonal polynomials. These are polynomials that the integral of a product of two of them is zero.

The set of orthog. poly. that we're interested is the <u>Legendre polynomials</u>, a collection $\{P_0, P_1, \cdots, P_n, \cdots\}$ that has the properties:

1. For each n, P_n is a polynomial of degree n
2. $\int_{-1}^{1} p(x) P_n(x)\, dx = 0$ whenever $p(x)$ is a poly of degree less than n.

The first few Legendre polynomials are

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=x^2-\frac{1}{3},\ P_3=x^3-\frac{3}{5}x,\ P_4(x)=x^4-\frac{6}{7}x^2+\frac{3}{35}$$

The roots of the poly lie in $(-1,1)$, have symmetry wrt origin

The nodes $x_1, x_2, \cdots, x_n$ needed to produce an integral approx formula to give exact results for any poly of degree less than $2n$ are the roots of the n^{th} degree Legendre poly.

Thm 4.7 Suppose $x_1, x_2, \cdots, x_n$ are roots of the n^{th} Legendre polynomial P_n and that for each $i = 1, 2, \cdots, n$, the numbers c_i are defined by $\int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x-x_j}{x_i - x_j}\, dx$.

If P is any poly of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i)$$

Proof: Case I: a polynomial of degree less than n, called $P(x)$.

We can rewrite (since it is unique) as an $(n-1)$st Lagrange polynomial with nodes at the roots of the n^{th} Legendre polynomial P_n. The representation is exact since the error term involves the n^{th} deriv of R (which is zero).

Hence,

$$\int_{-1}^{1} R(x)\,dx = \int_{-1}^{1}\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}R(x_i)\right]dx = \sum_{i=1}^{n}\underbrace{\left[\int_{-1}^{1}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}\right]}_{c_i}R(x_i)$$

$$= \sum_{i=1}^{n} c_i R(x_i)$$ $\Rightarrow$ result is true for poly of degree less than n.

If the Polynomial $P(x)$ of degree less than $2n$ is divided by nth Legendre polynomial $P_n(x)$, we get

$$P(x) = Q(x)P_n(x) + R(x)$$

It follows that

degree less than 2n   max deg n-1   degree n   degree less than n

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} Q(x)\,P_n(x) + R(x)\,dx$$

$$= \int_{-1}^{1} Q(x)\,P_n(x)\,dx + \int_{-1}^{1} R(x)\,dx$$

Since $P_n(x)$ is orthogonal to all polyn of degree less than n, then

$$= 0 + \underbrace{\int_{-1}^{1} R(x)\,dx}_{\text{done previously}}$$

$$= \sum_{i=1}^{n} c_i\,R(x_i) =$$

$$= \sum_{i=1}^{n} c_i P(x_i)$$

See table 4.11 for list of coeff & nodes.

what if we're not integrating between $[-1,1]$?

Then we do calculus witchcraft!

$$\int_a^b f(x)\,dx \underset{t=\frac{2x-a-b}{b-a}}{=} \underbrace{\int_{-1}^{1} f\left(\frac{(b-a)t+b+a}{2}\right)\frac{b-a}{2}\,dt}_{\text{apply gaussian quadrature to this.}}$$

Ex: $\int_1^{1.5} e^{-x^2}\,dx = .1093643$

Newton-Cotes

n	0	1	2	3	4
closed		.1183197	.1093104	.1093404	.1093643
open	.1048057	.1063473	.1094116	.1093971	

$$\int_1^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} \underbrace{e^{-\left(\frac{t+5}{4}\right)^2} \frac{1}{4}}_{\text{apply gaussian quadrature here.}} \cdot dt$$

$t = \frac{2x - 2.5}{.5} = 4x - 5 \;\rightarrow\; \frac{t+5}{4}$

$dt = 4\,dx$

<u>$n=2$</u>

$$\int_1^{1.5} e^{-x^2}\,dx = \frac{1}{4}\left[e^{-(5+.5773502692)^2/16} + e^{-(5-.5773502692)^2/16}\right]$$

$$= .1094003$$

<u>$n = 3$</u>

$$\int_1^{1.5} e^{-x^2}\,dx = \frac{1}{4}\Big[.555\bar{5}\, e^{-(5+.7745966692)^2/16} + .8\bar{8}\, e^{-(5)^2/16}$$
$$+ .\bar{5}\, e^{-(5-.7745966692)^2/16}\Big]$$

$$= .1093642$$

<u>Compare to Romberg</u>

.1183197			
.1115627	.1093104		
.1099114	.1093610	.1093643	
.1095009	.1093641	.1093643	.1093643