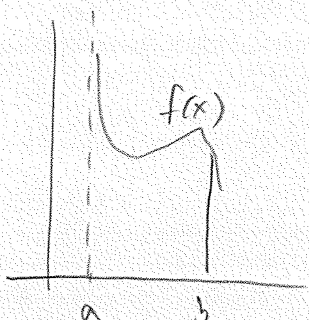
$\boxed{4.9}$ Improper Integrals

Two types: $\int^{\infty}$ ← top or $\int_0 f(x)\,dx$ ← when $f(x) \to \pm\infty$ as $x \to 0$.

First case when left endpoint is unbounded



Note! $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ converges iff $0 < p < 1$

and is equal to $\dfrac{(b-a)^{1-p}}{1-p}$

If f is a function that can be written as

$$f(x) = \frac{g(x)}{(x-a)^p}, \text{ where } 0<p<1, \text{ with } g \in C[a,b],$$

then $\int_a^b f(x)\,dx$ exists.

If $g \in C^5[a,b]$, then we construct 4^{th} Taylor Polynomial about $x=a$

$$P_4(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \frac{g'''(a)}{3!}(x-a)^3 + \frac{g^{(4)}(a)}{4!}(x-a)^4$$

$$= \sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!}(x-a)^k$$

and then write

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x) - P_4(x) + P_4(x)}{(x-a)^p}\,dx$$

$$= \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p} + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx.$$

Focus first on second part

$$(*)\quad \int_a^b \frac{P_4(x)}{(x-a)^p} = \sum_{k=0}^{4} \int_a^b \frac{g^{(k)}(a)}{k!}(x-a)^{k-p} = \underbrace{\sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!(k+1-p)}(b-a)^{k+1-p}}_{\text{exact answer to it.}}$$

Note that this is dominant part, especially when $P_4 \simeq g$

Next, we want to work on the first integral

$$\int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}\,dx$$

Define $G(x) = \begin{cases} \dfrac{g(x)-P_4(x)}{(x-a)^p}, & \text{if } a < x \le b \\ 0, & \text{if } x = a \end{cases}$

Note that since $0 < p < 1$ and $P_4^{(k)}(a)$ agrees with $g^{(k)}(a)$, $k = 0,1,2,3,4$,

then $G \in C^4[a,b] \Rightarrow$ we can use composite Simpson's rule

Then [illegible] on it, and error term will work. Add this answer to (*) and we have the approximation.

Ex: $\displaystyle\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\int_0^1 \frac{P_4(x)}{\sqrt{x}} = \int_0^1 \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{2} + \frac{x^{5/2}}{6} + \frac{x^{7/2}}{24}\,dx$$

$$= 2\sqrt{x} + \frac{2}{3}x^{3/2} + \frac{1}{5}x^{5/2} + \frac{2}{7\cdot 6}x^{7/2} + \frac{x^{9/2}}{9\cdot 12}\Bigg]_0^1$$

$$= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} = \frac{11051}{3780} \approx 2.92354497354$$

Next, we apply Simpson's Rule to

$$G(x) = \begin{cases} \dfrac{e^x - P_4(x)}{\sqrt{x}}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Using Simpson's Rule on G(x) ~~yields~~ w/100 pts yields

0.001758518296522 45

Adding to the other integral gives

$$\frac{11051}{3780} + \quad = 2.925303491841496$$

This gives the bound on the error of $(G^4(x) \le 1$ on $[0,1])$

(A) $$\frac{b-a}{180} h^4 f^4(u) \le \frac{1}{180}\left(\frac{1}{100}\right)^4 (1) = \underline{5.5 \times 10^{-11}}$$

10 digits accurate!

(B) If we have the singularity at the right endpoint, we do the same method

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(C) If we have one on both sides, we split into 2.

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\text{apply technique (A)}} + \underbrace{\int_c^b f(x)\,dx}_{\text{apply technique (B)}}$$

(D) $\int_a^{\infty} f(x)\,dx \Rightarrow$ do the substitution $t = \frac{1}{x} \Rightarrow dt = -\frac{1}{x^2}\,dx$

$$dx = -x^2\,dt$$
$$= -\frac{1}{t^2}\,dt$$

$$= \int_{1/a}^{0} f\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\,dt\right) = \underbrace{\int_0^{1/a} \frac{1}{t^2} f(t)\,dt}_{\text{apply (A)}}$$