[1.4] Some Properties of Probability

Thm 1.4.1

If A is an event and A' is its complement, then

$$P(A) = 1 - P(A')$$

Proof:

$$A \cup A' = S$$

So

$$P(A \cup A') = P(S)$$

$$P(A) + P(A') = 1$$

$$P(A) = 1 - P(A')$$

Ex: 3 births

$X = \#$ of girls

X	
0	BBB
1	BBG
	BGB
	GBB
2	BGG
	GBG
	GGB
3	GGG

It is clear that

$P(\text{At least one girl}) =$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

However, we can also find this using Complement Rule:

$A =$ At least one girl

$A' =$ no girls

$$P(A) = 1 - P(A') = 1 - \frac{1}{8} = \frac{7}{8}$$

Now imagine 50 kids! Much easier

Thm. 1.4.2 For any event A,

$$P(A) \le 1.$$

Proof: Easy! Suppose A' is the complement of A. Then, by axioms of prob,

$$P(A') \geq 0$$

So, since A' is the complement then $P(A') = 1 - P(A)$

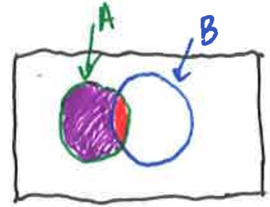
And

$$1 - P(A) \geq 0$$

So $P(A) \le 1$

Thm 1.4.3 For any events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$A = \underline{(A \cap B)} \cup \underline{(A \cap B')}$$

$$A \cup B = (A \cap B') \cup B$$

So $P(A) = P(A \cap B) + P(A \cap B')$

$$\Rightarrow \underline{P(A \cap B')} = P(A) - \underline{P(A \cap B)}$$

and

$$P(A \cup B) = \underline{P(A \cap B')} + P(B)$$

$$= P(A) - P(A \cap B) + P(B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$\boxed{\text{Thm } 1.4.4}$ For any three events,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof: Let $D = B \cup C$ and use 1.4.3 repeatedly

$$P(A \cup B \cup C) = P(A \cup D)$$

$$= P(A) + P(D) - P(A \cap D)$$

$$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Note: the pattern holds for larger numbers of sets.

$\Rightarrow$ Add in all 1 at a time

Subtract all 2 at a time

Add in all 3 at a time

+ etc

Thm 1.4.5 If $A \subseteq B$, then $P(A) \leq P(B)$

Thm 1.4.6 Boole's Inequality

$A_1, A_2, \ldots$ sequence of sets

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Thm 1.4.7 Bonferroni's Ineq.

$$P\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - \sum_{i=1}^{k} P(A_i')$$