[1.5] Conditional Probability

Numerous case of prob of an event A will be affected by knowledge of the occurrence of another event B. In these cases, the conditional prob of A given B is notated as $P(A|B)$

EX: Blood Types

	O	A	B	AB	
Rh+	114	102	27	9	252
Rh−	21	18	6	3	48
	135	120	33	12	300

$$P(B) = \frac{33}{300} = \frac{11}{100} = .11$$

$$P(B|Rh+) = \frac{27}{252} = \frac{n(B \,\&\, Rh+)}{n(Rh+)} = \frac{3}{28} = .107$$

In general (Not blood types):

DEF 1.5.1 $\quad P(A|B) = \dfrac{P(A \cap B)}{P(B)}$,if $P(B) \neq 0$

Note: conditional prob are prob! and rule the apply to prob, also apply to con prob. A_1 & A_2 are mutually exclusive, then

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$$

Show:

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1|B) + P(A_2|B)$$

Some other rules:

$$P(A|B) \geq 0$$

$$P(S|B) = P(B|B) = 1$$

$$P(A|B) = 1 - P(A'|B)$$

$$0 \leq P(A|B) \leq 1$$

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) \quad P(A_1 \cap A_2|B$$

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

In other words, conditional probabilities behave and follow the same rules that normal probabilities do. They are probabilities!

$\boxed{\text{Thm 1.5.1}}$ Multiplication Rule

For any two events A, B,

$$P(A \cap B) = P(B) \cdot P(A|B) = P(A)\ P(B|A)$$

This formula is used a lot when sample without replacement.

For example, suppose we draw two cards from a deck of standard playing cards

$$P(AK) = P(A) \cdot P(K|A)$$
$$= \frac{4}{52} \quad \frac{4}{51}$$

Total Probability & Bayes Risk

Sometimes it is useful to partition an event into the union of 2 or more disjoint events.

For example B & B' can be used to split A into:

$$A = (A \cap B) \cup (A \cap B')$$

In general, if $S = B_1 \cup B_2 \cup \cdot \cup B_k$, where B_k's are mutually exclusive (disjoint), then

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \quad \cup (A \cap B_k)$$

$\boxed{\text{Thm } 1.52}$ Law of Total Probability

If $B_1, B_2, \ldots, B_k$ is a collection of M.E. and exhaustive events, then, for any A,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)$$

Proof:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)$$

$$A = \bigcup_{i=1}^{k} A \cap B_i$$

Prob of union of disjoint events

$$P(A) = P\left[\bigcup_{i=1}^{k} A \cap B_i\right] = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)$$

Example Microchips.

Produced at two factories
Factory one has two shifts.

DEFINE:

B_1 = chip produced at factory 1 & shift 1.

B_2 = " " " " 1 " " 2.

B_2 = " " " " 2.

A = obtaining a defective chip.

Table:

	B_1	B_2	B_3	
A	5	10	5	20
A'	20	25	35	80
	25	35	40	100

Various Probs:

$P(B_1) = \frac{25}{100}$, $P(B_2) = \frac{35}{100}$, $P(B_3) = \frac{40}{100}$

$P(A) = \frac{20}{100}$

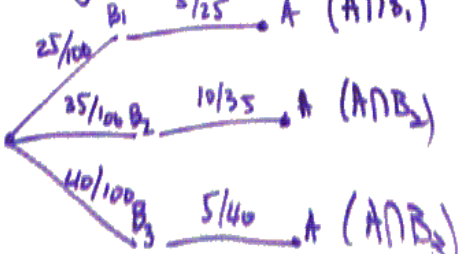
Note that $P(A)$ can also be found by:

$$P(A) = P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)$$

$$= \frac{25}{100}\left(\frac{5}{25}\right) + \frac{35}{100}\left(\frac{10}{35}\right) + \frac{40}{100}\left(\frac{5}{40}\right)$$

$$= \frac{20}{100}$$

Tree Diagram



Suppose the chips are sorted into 3 boxes:

Box 1 – 25 microchips from shift 1

Box 2 – 35 " " " 2

Box 3 = 40 " " factory 2

Box 1	Box 2	Box 3
5 def	10 def	5 def.
20 good	25 good	35 good

Experiment: Pick a box at random, then a chip at random.

$$P(A) = \sum_{i=1}^{3} P(\text{Box}_i)\cdot P(A|\text{Box}_i)$$

$$= \frac{1}{3}\cdot\frac{5}{25} + \frac{1}{3}\cdot\frac{10}{35} + \frac{1}{3}\cdot\frac{5}{40} = \frac{57}{280}$$

Independence:

Two Events A & B are called independent if $P(A \cap B) = P(A) \cdot P(B)$

Otherwise, they are dependent.

$\boxed{\text{Thm } 1.5.4}$ A, B events $P(A) > 0$, $P(B) > 0$

A & B are independent iff $P(A) = P(A|B)$ and $P(B) = P(B|A)$

$\boxed{\text{Thm } 1.5.5}$ A & B are independent iff the following pairs are independent

1. A and B'
2. A' and B
3. A' and B'

$\boxed{\text{DEF } 1.5.3}$

The k events $A_1, \dots, D_k$ are said to be mutually indep. If for every $j = 2, \dots, k$ and every subset of distinct indices.

$$i_1, i_2, \dots, i_j$$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_j})$$

Note: pairwise independence does not imply independence of 3 or more events.

New idea: Suppose the component obtained is defective, but it is not known which box it comes from. Then it is possible to compute the prob. that it came from a particular box given it was defective by using Bayes Rule formula.

Bayes Rule Thm 1.5.3. (Same conditions as 1.5.2)

$$P(B_j|A) = \frac{P(B_j)\cdot P(A|B_j)}{\sum P(B_i)\, P(A|B_i)}$$

Proof:

$$P(B_j|A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(B_j)\cdot P(A|B_j)}{\sum P(B_i)\cdot P(A|B_i)}$$

So, apply the rule to the problem at hand. Suppose we have a defective microchip, what is the prob. it came from Box 1?

$$P(B_1|A) = \frac{P(B_1)\cdot P(A|B_1)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)}$$

Suppose B_i = Box i

$$= \frac{(1/3 \cdot 5/25)}{\left(\frac{1}{3}\right)\left(\frac{5}{25}\right) + \left(\frac{1}{3}\right)\left(\frac{10}{25}\right) + \frac{1}{3}\left(\frac{5}{40}\right)} = \frac{56}{171} = 0.327485$$