[1.6] Counting Techniques

Multiplication Rule

If one operation can be performed n_1 ways

and a second " " " n_2 ",

then there are $n_1 n_2$ ways in which both can be performed together

Ex: How many diff't lunches could a person pick from Bob's Burger joint?

Bob serves 10 diff't burgers

3 sizes of fries

7 flavors of soda

$10 \cdot 3 \cdot 7 = \underline{\underline{210}}$

Thm 1.6.1 If there are N possible outcomes of each of r trials of an experiment, then there are N^r possible outcomes in the sample space.

Proof: (Mult. rule)

$$\underbrace{\underline{N}\ \underline{N}\ \underline{N}\ \underline{N} \cdots \underline{N}\ \underline{N}\ \underline{N}\ \underline{N}\ \underline{N}}_{r \text{ different}} = N^r$$

Ex: How many different ways could a multiple choice test be answered where there is 20 questions & 4 different ~~parts to each~~ question?

$$\underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4} \cdots \underline{4}\ \underline{4}\ \underline{4} = 4^{20}$$

$= 1{,}099{,}511{,}627{,}776$

In counting problems, the order of the selection of objects may or may not be important.

In addition, sometimes sampling is <u>without replacement</u>, or <u>with replacement</u>

Without replacement means an object can be selected at most once. Typically, we are refering to distinct objects (that can be handled).

<u>DEF</u>: An orderd arrangement of n distinguishable objects is known as a <u>permutation</u>

<u>Thm 1.6.2</u> The number of permutations of n distinguishable objects is $n!$

<u>Note</u>: the term indistinguishable mean that we cannot tell or we don't care about the difference between some objects. Distinguishable means they are all different.

$\boxed{\text{Thm 1.6.3}}$ The number of permutations of n objects taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example: □ ○ ◇

Pick 2 objects at a tim

$$\left.\begin{array}{ll} □○ & ○□ \\ □◇ & ◇□ \\ ○◇ & ◇○ \end{array}\right\} 6$$

If the order of

then one may be

of combinations

a combination a

of objects. (Y

and not the orde

En

Ex: □ ○ ◇

$$\left.\begin{array}{l} □○ \\ □◇ \\ ○◇ \end{array}\right\} \text{3 combinations}$$

$$_3C_2 = \frac{3!}{1!\,2!} = \frac{3\cdot\cancel{2\cdot 1}}{\cancel{2\cdot 1}} = 3$$

Thm 1.6.4 The number of combinations of n distinct objects chosen r at a time is
without replacement

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Indistinguishable objects

For example, suppose we want to see how many re-arrangements are possible of the word FOUR

There are 4 different letters, so there is $4! = 24$ different re-arrangements

What about the word SEEN?

It is useful to label the first E as E_1 & second as E_2.

Write out all the possible arrangements using E_1 & E_2 as distinct objects.

$S E_1 E_2 N$
$S E_1 N E_2$
$S N E_1 E_2$

$E_1 S E_2 N$
$E_1 S N E_2$
$E_1 N S E_2$

$N S E_1 E_2$
$N E_1 E_2 S$
$N E_1 S E_2$

$E_1 N E_2 S$
$E_1 E_2 N S$
$E_1 E_2 S N$

~~$S E_2 E_1 N$~~
~~$S E_2 N E_1$~~
~~$S N E_2 E_1$~~
~~$E_2 S E_1 N$~~
~~$E_2 S N E_1$~~
~~$E_2 N S E_1$~~
~~$N S E_2 E_1$~~
~~$N E_2 E_1 S$~~
~~$N E_2 S E_1$~~
~~$E_2 N E_1 S$~~
~~$E_2 E_1 N S$~~
~~$E_2 E_1 S N$~~

Now, drop the E_1 & E_2 to just E.

There are 12 distinct permutations.

Thm 1.6.5 The number of distinguishable permutations of n objects of which r are of one kind and $n-r$ are of the other kind is

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Note: the setting is different here! even though it is the same formula as the combinations rule

Thm: The number of distinguishable permutations of n objects of which r_1 are of one kind,
r_2 are of 2^{nd} kind,
$\vdots$
r_k are of k^{th} kind,

is

$$\frac{n!}{r_1!\,r_2!\cdots r_k!}$$

Note: $r_1 + r_2 + \cdots + r_k = n$

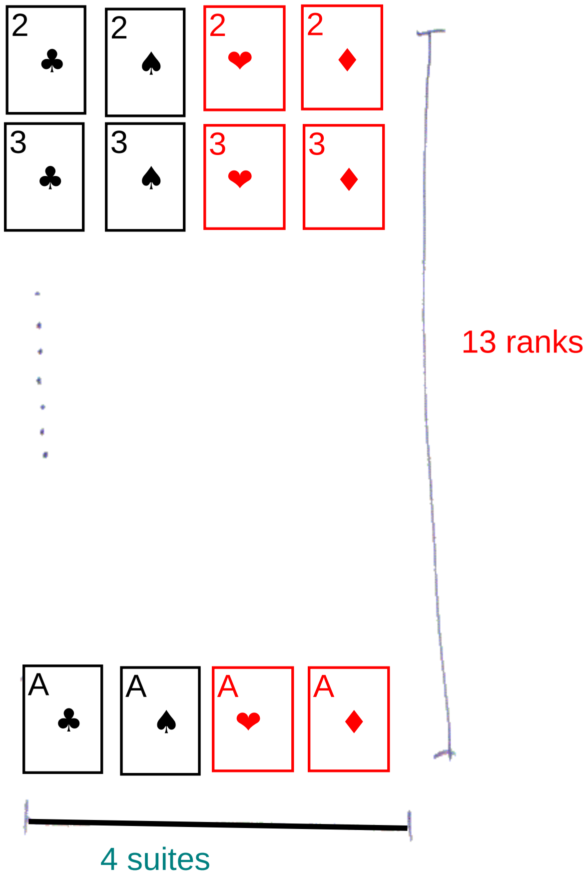
Ex: How many different rearrangements of the word MISSISSIPPI are there?

$$\frac{11!}{1!\,4!\,4!\,2!} = 34,650$$

Probability

$$P(\text{event } A) = \frac{n(A)}{N}$$

$\leftarrow$ both are computed using couting.



Count the number of Royal Flush → same suite.
└ royalty

A, K, Q, J, 10 — H, D, S, C ⇒ 4 royal flushes.

Straight

2,3,4,5,6
3,4,5,6,7
⋮
9,10,J,Q,K
10,J,Q,K,A

$\underbrace{9 \cdot 4^5}$ this includes straight flushes.

In total, you can pick 52 card 5 at a time:

$${}_{52}C_5 = \frac{52!}{5!\,47!} = 2,598,960$$

Full-House

3 kind w/ 2 kind.

A A A   K K

$13 \cdot {}_4C_3 \cdot 12 \cdot {}_4C_2 = 3,744$

↑ choose rank

$$P(\text{full house}) = \frac{3744}{2598960}$$