[2.1-2.2] Intro to Random Variables

DEF 2.1.1 A random variable, say X, is a function defined over a sample space S, that associates a real number with each possible outcome e in S.

$X(e) = x$

capital   little

Capital letters indicate random Variables (X, Y, Z)

lowercase letters indicate possible values the R.V can attain.

EX: Flip a coin 3 times

Sample space: $S = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$

$X = $ # of heads in 3 tosses.

$X(HHH) = 3$ $X(TTH) = 1$

$X(HTH) = 2$ $X(TTT) = 0$

Ex: Roll 2 dice

1	2	3	4	5	6
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$\boxed{X}$ = maximum of 2 dice → $X(5,4) = 5$

X = sum of two dice $X(5,4) = 9$

X = min of the 2 dice

examples of other possible random variables.

Note that in between the lines on the left, the elements are composed of the maximum of 2 dice.
For example the highlighted boxes all have a maximum value of 5

$\boxed{2.2}$ Discrete R.V.

DEF: If the set of all possible values of a R.V X, is a countable set (Note: could be infinite)

$x_1, x_2, \dots, x_n$ or $x_1, x_2, \dots$

then X is called a discrete R.V. The function

$$f(\overset{\text{small}}{x}) = P[\overset{\text{big}}{X} = \overset{\text{small}}{x}], \quad x = x_1, x_2, \dots$$

that assigns prob. to each possible x will be called the discrete prob. density function (pdf) (also caled pmf = prob. mass function)

Some time we write $f_X(x)$ instead of $f(x)$

A discrete RV can be displayed as a table, a graph, or a formula. Each of them identifies the distribution uniquely.

Thm 2.2.1

A function $f(x)$ is a discrete pdf iff it satisfies both of the following for at most a countably infinite set of real #'s

1) $f(x_i) \geq 0$ for all x_i

2) $\sum_{\text{all } x_i} f(x_i) = 1$

proof. follows from axioms of prob.

Homework!

Tables

Ex: Flip a coin 3 times.

$S = \{HHH, TTH, THT, HTT, THH, HTH, HHT, TTT\}$

X = # of Heads in 3 tosses

x	$f(x)$
0	$\frac{1}{8}$
1	$3/8$
2	$3/8$
3	$\frac{1}{8}$

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Ex: x = Max of two dice

x	$f(x)$
1	
2	
3	
4	
5	
6	

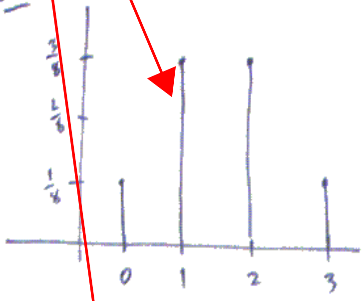
x	1	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

You can display the tables for the discrete pdfs either horizontally or vertically: I find horizontal nice when I have fractions and vertical for decimal numbers.

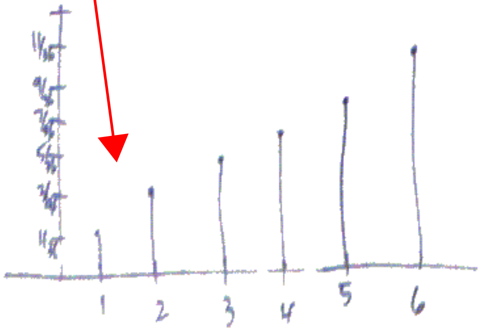
Graphs

Formula

Ex:



$$f(x) = {}_3C_x (.5)^3$$



formulas:

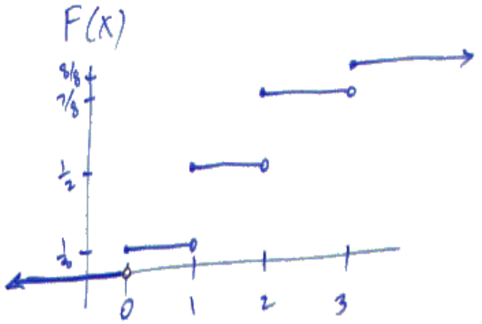
You can also write f(x) as a formula:

$$f(x) = \frac{2x-1}{36}, \quad x = 1, \ldots, 6$$

$\boxed{22}$ Cont.

DEF. The Cumulative dist function (CDF) of a random var X is defined for any real number x by

$$F(x) = P[\overset{\text{big}}{X} \le \underset{\text{small}}{x}]$$



often, we refer to the CDF as the distribution function.

For example,

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\Rightarrow$

x	0	1	2	3
$F(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$

Notation:

$$X \sim f(x) \quad \text{or} \quad X \sim F(x)$$

means that X has the pdf $f(x)$ and CDF $F(x)$

Note the $F(X)$ is a nondecreasing step function

Thm 2.2.2 Let X_i be a discrete R.V. with pdf $f(x)$ and CDF $F(x)$. If the possible values of X are indexed in increasing order

$$x_1 < x_2 < x_3 < \quad < x_n < \cdots$$

$$f(x_1) = F(x_1)$$

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Furthermore, if $x < x_1$, then $F(x) = 0$ and for any other real x,

$$F(x) = \sum_{x_i \le x} f(x_i)$$

Thm A function $F(x)$ is a CDF for some R.V. iff it satisfies the following

1. $\lim_{x \to -\infty} F(x) = 0$
2. $\lim_{x \to \infty} F(x) = 1$
3. $\lim_{x \to 0^+} F(x+h) = F(x)$ (Right continuous)
4. If $a < b$, then $F(a) \le F(b)$

$\boxed{\text{DEF 2.2.3}}$ If X is a discrete R.V with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \sum_{x} x f(x)$$

Expected value is also called the expectation or the mean

Ex: Expected value of

$X = \#$ of heads in 3 tosses.

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(x) = \sum_x x f(x) = 0\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) + 3\left(\tfrac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = 1.5$$

Ex: $X =$ max of two dice

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$E(x) = 1\left(\tfrac{1}{36}\right) + 2\left(\tfrac{3}{36}\right) + 3\left(\tfrac{5}{36}\right) + 4\left(\tfrac{7}{36}\right) + 5\left(\tfrac{9}{36}\right) + 6\left(\tfrac{11}{36}\right)$$

$$= \frac{161}{36} = 4.4722$$