[2.3] Continuous RVs (Random Variables)

DEF: A random variable X is called a continuous R.V if there is a function f(x), called the pdf of X, such that the CDF can be represented as

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

same x value. IMPORTANT!

Note $f(x) = \frac{d}{dx} F(x) = F'(x)$

Further, Note the def. of F(x)

$$F(x) = P[X \leq x]$$

$$P[a < X \leq b] = P[X \leq b] - P(X \leq a)$$
$$= F(b) - F(a)$$

Also,

$$P[a \leq X \leq b] = P(X \leq b) - P(X < a)$$
$$= P(X \leq b) - P(X \leq a) + P[X = a]$$
$$= F(b) - F(a) + \underbrace{P[X = a]}_{\text{what is this?}}$$

This term exists for discrete RVs, but disappears for continuous RVs

For a continuous probability distribution,

$P[X=c]=0$, Hence,

$P[a<X<b]=P[a\leq X<b]=P[a<X\leq b]=P(a\leq X\leq b)$

Thm: 2.3.1

A function $f(x)$ is a pdf for some continuous R.V. X iff

1) $f(x)\geq 0$ for all $x\in(-\infty,\infty)$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

integrate here!

Compare to discrete:

1) $f(x_i)\geq 0$ for all x_i

2) $\sum_{\text{all } i} f(x_i) = 1$

sum here!

Ex: Suppose $f(x)=\begin{cases} c(1+x)^{-3}, & x>0 \\ 0, & x\leq 0 \end{cases}$

where c is a constant.

By prop 1, $c>0$ in order for $f(x)\geq 0$.

By prop 2,

$$1=\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} c(1+x)^{-3}\,dx$$

$$=\lim_{b\to\infty}\left[\frac{c(1+x)^{-2}}{-2}\right]_0^b$$

$$=c\lim_{b\to\infty}\frac{1}{-2(1+b)^2}+\frac{c}{2}$$

$$=\frac{c}{2}=1 \Rightarrow \boxed{c=2}$$

Hence

$$f(x) = \begin{cases} 2(1+x)^{-3}, & x>0 \\ 0, & x \le 0 \end{cases}$$

The cdf for x is

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\,dt$$

Suppose $x \le 0$, then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

Suppose $x > 0$, then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{0} + \int_{0}^{x} f(t)\,dt$$

$$= \int_{0}^{x} f(t)\,dt = \int_{0}^{x} 2(1+t)^{-3}\,dt = \left.\frac{\cancel{2}(1+t)^{-2}}{-\cancel{2}}\right]_{0}^{x}$$

$$= 1 - \frac{1}{(1+x)^2}$$

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1 - \frac{1}{(1+x)^2}, & x > 0 \end{cases}$$

Note: $f(x)$ is not a probability, although it can be used to assign probability to arbitrarily small intervals.

Note: $P[a \le x \le b] = \int_a^b f(x)\,dx = \underset{\text{CDF}}{F}(b) - F(a)$

DEF: If X is cont with pdf $f(x)$, the expected value of X is

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

if the integral is absolutely convergent. Otherwise, we say the $E(x)$ does not exist.

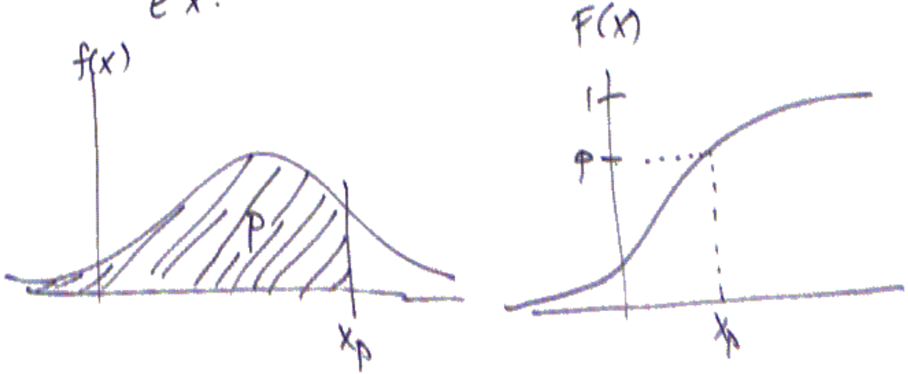
Compare to the discrete RV case. Instead of an integral like above, it uses a summation instead.

$\boxed{\text{DEF } 2.3.3}$

If $0 < p < 1$, then a $100 \times p^{th}$ percentile of the distribution of a continuous R.V X is a solution x_p to the equation

$$F(x_p) = p$$

Ex:



We can also think of this in terms of quantiles (e.g. the 97^{th} percentile is the .97 quantile)

The median of the dist X is a 50^{th} percentile, denoted by $X_{.5}$ or m.

Ex X is the dist of lifetimes (in months) of a particular component

the CDF is $F(x) = 1 - e^{-\left(\frac{x}{2}\right)^2}$, $x > 0$

Let's find the median of this random variable.
Start on the next page:

$\boxed{2.3}$ Cont

Find the median of $F(x) = 1 - e^{-\left(\frac{x}{3}\right)^2}$, $x > 0$.

Median is where $F(x_{.5}) = .5$

$$1 - e^{-\left(\frac{x_{.5}}{3}\right)^2} = .5$$

$$1 - .5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$.5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$\ln(.5) = -\left(\frac{x_{.5}}{3}\right)^2$$

$$3\sqrt{-\ln(.5)} = x_{.5} \quad (\text{since } x_{.5} > 0)$$

$$x_{.5} = 3\sqrt{\ln 2} = 2.4977$$

$\boxed{\text{Def:}}$ If the pdf has a unique max at $x = m_o$,
(e.g. $\max_{\text{all } x} f(x) = f(m_o)$), then m_o is called the mode of x

Ex: The pdf of the previous example is

$$f(x) = \left(\frac{2}{9}\right) x e^{-\left(\frac{x}{3}\right)^2}, \quad x > 0$$

$$f'(x) = 0 = \frac{2}{9}\left[x e^{-\left(\frac{x}{3}\right)^2}\left(-\frac{2}{9}x\right) + e^{-\left(\frac{x}{3}\right)^2}\right]$$

$$= \frac{2}{9} e^{-\left(\frac{x}{3}\right)^2}\left[-\frac{2}{9}x^2 + 1\right] = 0$$

$$-\frac{2}{9}x^2 + 1 = 0 \Rightarrow x^2 = \frac{9}{2}$$

$$x = \frac{3}{\sqrt{2}} = 2.121 \text{ months}$$

In general, the mean, median, and mode are all different, but there are cases when they agree.

DEF A dist with pdf $f(x)$ is said to be symmetric about c if

$$f(c-x) = f(c+x) \quad \text{for all } x.$$

Asymmetric distributions are called skewed dist.

Mixed Distribution

It is possible to have a random variable whose dist. is neither purely discrete nor continuous

A prob dist. for a RV X is of mixed type if the CDF has the form

$$F(x) = a F_d(x) + (1-a) F_c(x)$$

F_d is the CDF of a discrete R.V and F_c is the CDF of a continuous R.V. and $0 < a < 1$