[2.5] Moment Generating Functions

DEF 2.5.1 If X is a R.V., then the expected value

$$\boxed{M_X(t) = E[e^{tX}]}$$

is called the moment generating function (mgf) of X if the expected value exists for all values of t in some interval of the form $-h < t < h$ for some $h > 0$.

Sometimes the X is dropped from the formula and we write M(t) instead. Use whenever it is necessary.

in other words, it exists in a neighborhood of 0

Ex: Assume X is discrete and finite valued. In general, this is not necessary (it can be continuous too)

The mgf is

$$M_X(t) = E\left[e^{tX}\right] = \sum_{i=1}^{m} e^{tx_i} f(x_i)$$

The first derivative of $M_X(t)$ wrt t is

$$M_X'(t) = \sum_{i=1}^{m} x_i e^{tx_i} f(x_i)$$

The r^{th} derivative of $M_X(t)$ wrt t is

$$M_X^{(r)}(t) = \sum_{i=1}^{m} (x_i)^r e^{tx_i} f(x_i)$$

Note:

$$M_X^{(r)}(0) = \sum_{i=1}^{m} x_i^r f(x_i) = E\left[X^r\right] = \mu_r'$$

Thm 2.5.1 If the mgf exists, then

$$\mu_r' = E(X^r) = M_X^{(r)}(0) \quad \text{for } r = 1, 2, \cdots$$

and

$$\boxed{M_X(t) = \sum_{r=0}^{\infty} \frac{E(X^r)}{r!} t^r}$$

Ex: Consider the cont. R.V. X with pf

$$f(x) = \frac{1}{4} x e^{-x/2}, \quad x > 0$$

The mgf is

$$M_X(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx} \frac{1}{4} x e^{-x/2} dx$$

$$= \frac{1}{4} \int_0^\infty x e^{(t-\frac{1}{2})x} dx = \frac{1}{4} \int_0^{-\infty} e^u \frac{du}{t-\frac{1}{2}} \left(\frac{u}{t-1/2} \right.$$

$u = (t-\frac{1}{2})x = tx - \frac{1}{2}x$

$du = (t-\frac{1}{2})dx$

$dx = \frac{du}{\ldots}$

$t < \frac{1}{2}$

$$= \frac{1}{4(t-\frac{1}{2})^2} \int_0^{-\infty} u e^u \, du$$

$u = u \qquad dv = e^u du$

$du = du \qquad v = e^u$

$$= \frac{1}{4(t-\frac{1}{2})^2} \left[u e^u - \int e^u \, du \right]_0^{-\infty}$$

$$= \frac{1}{4(t-\frac{1}{2})^2} \underbrace{\left[u e^u - e^u \right]_0^{-\infty}}_{1}$$

$$= \frac{1}{4\left(\frac{2t-1}{2}\right)^2} = \frac{1}{(2t-1)^2} = \left(\frac{1}{1-2t}\right)^2$$

$, t < 1/2$

Ex: A discrete r.v. with pdf

$$f(x) = .7^x \, , \; x = 1, 2, \ldots$$

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}(0.7)^x$$

$$= \sum_{x=1}^{\infty} (0.7e^t)^x = \frac{0.7e^t}{1-0.7e^t} = \frac{7e^t}{10-7e^t}$$

(Note: the mgf will exist as long as $|0.7e^t| \leqslant 1$

$$0 < (0.7e^t) < 1 \Longrightarrow 0 < e^t < \frac{10}{7} \Longrightarrow t < \ln(10/7)$$

$\boxed{2.5}$ Cont

Properties of Moment Generating Functions

$\boxed{\text{Thm } 2.5.2}$

If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$

Proof:

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] =$$

$$= E[e^{tb} e^{X(at)}] = e^{tb} \underbrace{E[e^{X(at)}]}_{M_X(at)}$$

$$= e^{tb} M_X(at)$$

It can be shown the MGF's uniquely determine a distribution (if MGF exists)

$\boxed{\text{Application of Thm } 2.5.2}$

$Y = X - \mu$

Then

$$E[(X-\mu)^r] = \frac{d^r}{dt^r}\left[e^{-\mu t} M_X(t)\right]\Big|_{t=0}$$

$\boxed{\text{Thm } 2.5\ 3}$ Uniqueness

If X_1 & X_2 have respective CDFs $F_1(x)$ and $F_2(x)$ and MGFs $M_1(t)$ and $M_2(t)$

then $F_1(x) = F_2(x)$ for all real x iff

$M_1(t) = M_2(t)$ for all t in some neighborhood of 0

($-h < t < h$ for some $h > 0$)

Factorial Moments

DEF 2.52

The r^{th} factorial moment of X is

$E[X(X-1)\cdots(X-r+1)]$ and the factorial moment generating function (FMGF) of X is

$$G_X(t) = E[t^X]$$

if this expectation exists for all t in a neighborhood of 1 $(1-h < t < 1+h$ and $h > 0)$

Note that the Factorial moment generating function must exist in a neighborhood of 1 (not 0 like for a normal MGF).

The factorial moment generating function is also called the prob. generating function because for non-negative integer valued random var. X,

$$P[X=r] = \frac{G_X^{(r)}(0)}{r!}$$

Evaluating at zero here is not a misprint! This will allow you to generate probabilities

Note the relationship:

$$G_X(t) = E[t^X] = E[e^{\ln t^X}] = E[e^{(\ln t)X}] = M_X(\ln t)$$

$\boxed{\text{Thm 2.5.4}}$ If X has FMGF, $G_X(t)$, then

$$G_X'(1) = E(X)$$

$$G_X''(1) = E[X(X-1)]$$

$$\vdots$$

$$G_X^{(r)}(1) = E[X(X-1)\cdots(X-r+1)]$$

Note that regular moments can be derived from the factorial moments. For example:

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$\Rightarrow E(X^2) = E[X(X-1)] + E(X)$$

$$= G_X''(1) + G_X'(1)$$

You just need to be willing to do the algebra to solve for the others. Reasons for doing it usually only occur with discrete distributions.