[3.3] Special Continuous Distributions

We will study several dist.

Uniform
Gamma, Beta
Exponential
Weibull
Pareto
Normal

Uniform

Suppose that a cont. R.V. can assume values only in a bounded interval, say $a < x < b$, and suppose that $f(x)$ is constant over that interval

Then $X \sim \text{UNIF}(a, b)$

$$f(x; a, b) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases} = \frac{1}{b-a} I_{(a,b)}(x)$$

Where $I_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$ and is called the indicator function

It "indicates" where the function is "on", like a light switch

The CDF is

$$F(x; a, b) = \begin{cases} 0, & x \leqslant a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geqslant b \end{cases}$$

$$= \frac{x-a}{b-a} I_{(a,b)}(x) + I_{[b,\infty)}(x)$$

The expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{2} \frac{b^2-a^2}{b-a}$$

$$= \frac{1}{2} \frac{(a+b)(\cancel{b-a})}{\cancel{b-a}} = \frac{a+b}{2}$$

Gamma Distribution

κ (kappa) (greek letter)

DEF 3.31 - Gamma Function

The gamma function, denoted by $\Gamma(\kappa)$ for all $\kappa > 0$, is given by

$$\Gamma(\kappa) = \int_0^\infty t^{\kappa-1} e^{-t} dt$$

Captial Gamma (like a hangman stand!

The gamma function satisfies the following

$$\Gamma(\kappa) = (\kappa - 1)\Gamma(\kappa - 1), \ \kappa > 1$$
$$\Gamma(n) = (n-1)!, \ n \in \mathbb{N}$$
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

This is really, really strange! What does pi have to do with factorials?!!

Hint: To prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, write $[\Gamma(\frac{1}{2})]^2 = \pi$ as a double Integral (after subsituting $t = u^2$) and change to polar coordinates.

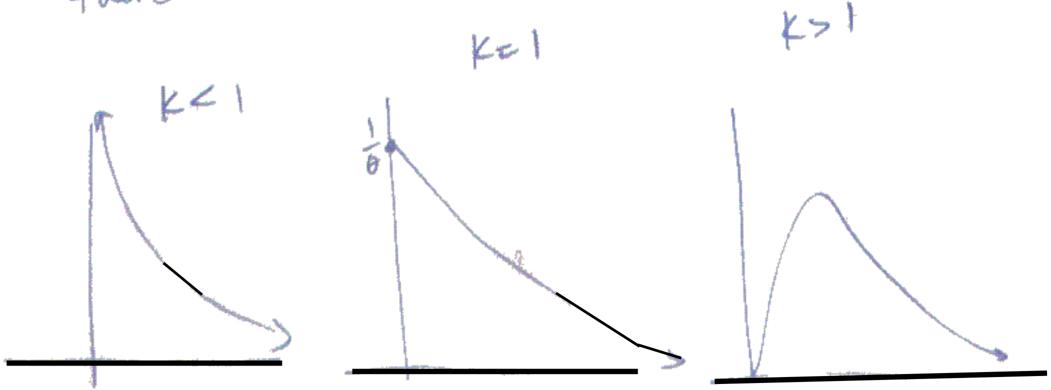
A contin. R.V. X is said to have the gamma dist with param $\kappa > 0$, and $\theta > 0$ if

$$f(x; \theta, \kappa) = \frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-x/\theta} I_{(0,\infty)}(x)$$

Note: $X \sim GAM(\theta, \kappa)$

The parameter κ is also called a shape parameter because it determines the base shape of the dist. Specifically, there are three different shapes

Shape (κ) Parameter



The CDF of the gamma is

$$F(x;\theta,\kappa)=\begin{cases}0, & x\leqslant 0\\ \displaystyle\int_0^x \frac{1}{\theta^\kappa\Gamma(\kappa)}t^{\kappa-1}e^{-\frac{t}{\theta}}\,dt, & x>0\end{cases}$$

The θ parameter is called the Scale parameter. This is important because you don't want your results to depend on scale of measurement used.

A Scale parameter satisfies the relation

$$F(x;\theta)=F\left(\frac{x}{\theta}\right)$$

The CDF $F(x;\theta,\kappa)$ cannot be generally solved for. However, when κ is an integer, then it can. (This is equiv to doing tab integ

Thm 3.3.2 If $X\sim GAM(\theta,n)$, where n is an integer and if $Y\sim POI(\frac{x}{\theta})$

$$P[X<x]=P[Y\geqslant n]=1-\sum_{i=0}^{n-1}\frac{(x/\theta)^i}{i!}e^{-x}$$

The Mean of the Gamma Distribution

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \left(\frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-x/\theta} \right) dx$$

$$u = \frac{x}{\theta} \Longrightarrow x = u\theta$$

$$du = \frac{1}{\theta} dx \Longrightarrow dx = \theta du$$

$$= \int_0^{\infty} u\theta \left(\frac{1}{\theta^\kappa \Gamma(\kappa)} u^{\kappa-1} \theta^{\kappa-1} e^{-u} \right) \theta du$$

$$= \frac{\theta}{\Gamma(\kappa)} \int_0^{\infty} u^\kappa e^{-u} du = \frac{\theta}{\Gamma(\kappa)} \underbrace{\int_0^{\infty} u^{(\kappa+1)-1} e^{-u} du}_{\Gamma(\kappa+1)}$$

$$= \frac{\theta \Gamma(\kappa+1)}{\Gamma(\kappa)} = \frac{\theta \kappa \Gamma(\kappa)}{\Gamma(\kappa)} = \kappa\theta$$

General Moments:

$$E(X^r) = \int_0^{\infty} x^r \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta} dx$$

$$= \frac{1}{\theta^k \Gamma(k)} \underbrace{\int_0^{\infty} x^{r+k-1} e^{-x/\theta} dx}_{\text{Kernal of Gamma}(\theta,\, r+k)}$$

$$= \frac{1}{\theta^k \Gamma(k)} \frac{\theta^{k+r} \Gamma(k+r)}{1} \underbrace{\int_0^{\infty} \frac{1}{\theta^{k+r} \Gamma(k+r)} x^{r+k-1} e^{-x/\theta} dx}_{\text{integrates to 1 (pdf)}}$$

$$= \frac{\theta^{k+r} \Gamma(k+r)}{\theta^k \Gamma(k)}$$

$$= \frac{\theta^r \Gamma(k+r)}{\Gamma(k)}$$

$\boxed{3.3}$ Cont.

The MGF of Gamma dist is

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^k$$

Special Cases

If $\theta = 2$, $k = \nu/2$, where ν = degrees of freedom,

then $X \sim \chi^2(\nu)$ (chi-squared dist)

Exponential Distribution

If $k=1$, then GAMMA dist is the EXPONENTIAL dist

If $k=1$, then $X \sim EXP(\theta)$

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0,\infty)}(x)$$

$$F(x) = (1 - e^{-x/\theta}) I_{(0,\infty)}(x)$$

Note that θ is a scale parameter.

The exp dist. is useful as a prob. model for lifetimes.

The No-Memory property

Thm 3.3.3. For a cont. R.V. X, $X \sim EXP(\theta)$ iff

$$P[X > a+t \mid X > a] = P[X > t]$$ for all $a > 0$ and $t > 0$.

Think of it as this: An old component that still works is just as reliable as a new compon

Weibull Distribution

(A lot like Gamma Dist)

Used for:
- fatigue & breaking strength of materials
- failure times
- Engineers love it!
- CDF is explicit!

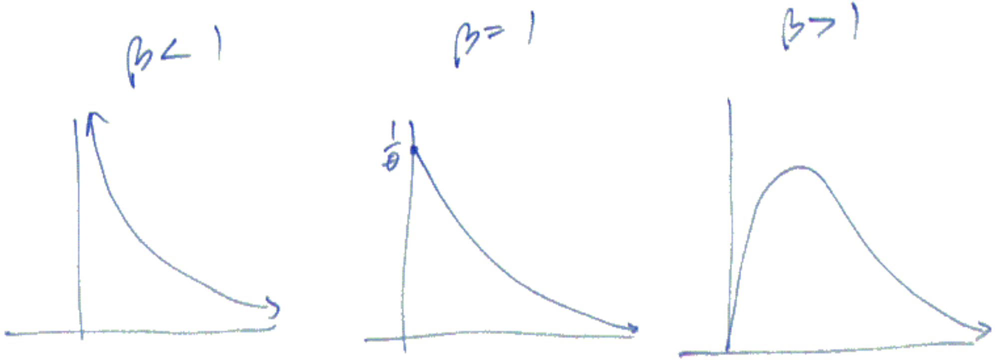
$$X \sim \mathrm{WEI}(\theta, \beta)$$

$$f(x;\theta,\beta) = \frac{\beta}{\theta^\beta} x^{\beta-1} \exp\left[-\left(\frac{x}{\theta}\right)^\beta\right] I_{(0,\infty)}(x)$$

β = shape parameter

θ = scale parameter

Shapes: **Shape Parameter (β)**



Note the similarities to the Gamma Dist!

CDF is $F(x;\theta,\beta) = 1 - e^{\left[-\left(\frac{x}{\theta}\right)^\beta\right]} I_{(0,\infty)}(x)$

Special Cases

If $\beta = 1$, then $X \sim \mathrm{EXP}(\theta)$ exponential again!

If $\beta = 2$, then $X \sim \mathrm{WEI}(\theta, 2)$ and we call it the Rayleigh dist

The mean of the Weibull is

$$\boxed{E(x) = \theta\Gamma\left(1+\frac{1}{\beta}\right)}$$ valid if $1+\frac{1}{\beta}>0$

$\Rightarrow$ always satisfied since $\beta>0$

The mgf does not exist in a form that is useful.

The variance is

$$V(x) = \boxed{\theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]}$$

The $100 \times p^{th}$ percentile is

$$\boxed{x_p = \theta\left[-\ln(1-p)\right]^{1/\beta}}$$

Pareto Distribution

$X \sim PAR(\theta, \kappa)$ κ = shape param. θ = scale param.

$$f(x;\theta,\kappa) = \left(\frac{\kappa}{\theta}\right)\left(1+\frac{x}{\theta}\right)^{-(\kappa+1)} I_{(0,\infty)}(x)$$

$\kappa > 0$.

CDF: $F(x;\theta,\kappa) = \left[1-\left(1+\frac{x}{\theta}\right)^{-\kappa}\right] I_{(0,\infty)}(x)$

Useful for modeling length of a wire between flaws: Ex: 2.3.2 is an example.

Note the pdf $f(y) = \left(\frac{\kappa}{a}\right)\left(\frac{y}{a}\right)^{-(\kappa+1)} I_{(a,\infty)}(y)$ is also called a pareto dist.

The mean & variance are

$$E(x) = \frac{\theta}{\kappa-1}, \kappa > 1 \quad V(x) = \frac{\theta^2\kappa}{(\kappa-2)(\kappa-1)^2}, \kappa > 2$$

pth percentile $x_p = \theta[(1-p)^{(-1/k)} - 1]$

Normal Distribution

First published in 1733 as an approx for the sum of Binomial random vars

It is the single most important dist in stats & prob.

$$f(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \sigma > 0$$

Simply it is denoted by $X \sim N(\mu, \sigma^2)$

It is also called the Gaussian dist

Verify it integrates to 1.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \frac{1}{\sqrt{\pi}}\underbrace{\left[2\int_0^{\infty} e^{-u^2}\,du\right]}_{\Gamma(\frac{1}{2})} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$z = \dfrac{x-\mu}{\sigma}$

$dz = \dfrac{dx}{\sigma}$

$u = \dfrac{t}{\sqrt{2}}$

$du = \dfrac{1}{\sqrt{2}}\,dt$

$\phi = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal pdf

$$\Phi(z) = \underbrace{\int_{-\infty}^{z} \phi(t)\,dt}_{\text{Standard normal CDF}}$$

Note: $\phi(z) = \phi(-z)$ (Even)

unique max at $z=0$ & inflection pts at ± 1

$\phi'(z) = -z\,\phi(z)$

$\phi''(z) = (z^2-1)\,\phi(z)$

$E(z) = 0$

$E(z^2) = 1 = V(z)$

Normal Dist cont.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{d(-z^2/2)}{dz} = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\phi(z)} (-z) = -z\phi(z)$$

$$\phi''(z) = \frac{d(-z\phi(z))}{dz} = -z\phi'(z) + \phi(z)(-1)$$

$$= -z(-z\phi(z)) - \phi(z)$$

$$= (z^2-1)\phi(z)$$

$$z^2\phi(z) = \phi''(z) + \phi(z)$$

$$\phi'''(z) = (z^2-1)\phi'(z) + 2z\,\phi(z)$$

$$= (z^2-1)(-z\phi(z)) + 2z\,\phi(z)$$

$$= (-z^3 + 3z)\phi(z)$$

$$= z(-z^2+3)\phi(z)$$

Max of $\phi(z)$ is $\phi'(z) = 0$

$$-z\phi(z) = 0 \Rightarrow z = 0$$

Inflection pts: $\phi''(z) = 0$

$$(z^2-1)\phi(z) = 0$$

$$z^2 = 1 \Rightarrow z = \pm 1$$

$$E(z) = \int_{-\infty}^{\infty} z\,\phi(z)\,dz = -\int_{-\infty}^{\infty} \phi'(z)\,dz$$

$$= -\lim_{b\to\infty}\Big[\phi(z)\Big]_{-b}^{b} = \lim_{b\to\infty} \phi(-b) - \phi(b)$$

$$= 0 - 0 = 0$$

$$E(z^2) = \int_{-\infty}^{\infty} z^2\phi(z)\,dz = \int_{-\infty}^{\infty} \phi''(z) + \phi(z)\,dz$$

$$= \int_{-\infty}^{\infty} \phi''(z)\,dz + \int_{-\infty}^{\infty} \phi(z)\,dz = \Big[\phi'(z)\Big]_{-\infty}^{\infty} + 1 = 1$$

Thus, $V(Z) = E(Z^2) - [E(Z)]^2$

$= 1 - 0$

$= 1$

Thm 3.3.4

If $X \sim N(\mu, \sigma^2)$, then

1. $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

2. $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

standard normal cumulative probabilities are in Table 3 In Appendix (in your book

Because of symmetry, only positive z values are given For negative z-values,

$$\Phi(-z) = 1 - \Phi(z)$$

Ex: $\Phi(z) = .9772$

Let z_γ denote the γ^{th} percentile of the standard normal, which means

$$\Phi(z_\gamma) = \gamma$$

For example, $\Phi(z_{.90}) = .90 \Rightarrow z_{.90} = 1.282$

It is often useful to consider normal prob in terms of standard deviations from the mean.

$$P[\mu - 2\sigma < X < \mu + 2\sigma] = F_X(\mu+2\sigma) - F_X(\mu - 2\sigma)$$

$$= \Phi\left(\frac{\mu+2\sigma-\mu}{\sigma}\right) - \Phi\left(\frac{\mu-2\sigma-\mu}{\sigma}\right) = \Phi(2) - \Phi(-2)$$

$$= 0.9772 - (1 - .9772)$$

$$= 0.9544$$

Note that the normal random variable can still be used as a reasonable model for a random variable that takes on only positive values, if very little prob. is associated with the neg. values.
(Another choice is to use a truncated normal model.)

Thm 3.3.5

$\boxed{\begin{array}{l}\text{If } X \sim N(\mu, \sigma^2), \text{ then} \\ (1)\ M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\} \\ (2)\ E(X-\mu)^{2r} = \dfrac{(2r)!\ \sigma^{2r}}{r!\ 2^r}, \quad r = 1, 2, \ldots \\ (3)\ E(X-\mu)^{2r-1} = 0 \qquad, r = 1, 2, \ldots\end{array}}$

Proof:

(1) Complete the square in the exponent and use the "kernel method".

(2)
(3) Expand the MGF in a Maclaurin series.