[4.3] Joint Continuous Distributions

$\boxed{\text{DEF:}}$ a k-dimensional vector-valued random variable $X = (X_1, X_2, \ldots, X_k)$ is said to be **continuous** if there is a function $f(x_1, x_2, \ldots, x_k)$ called the **joint probability density function** (joint pdf), of X, such that the joint CDF can be written as:

$$F(x_1, \ldots, x_k) = \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f(t_1, \ldots, t_k) dt_1 \cdots dt_k$$

for all $x = (x_1, \ldots, x_k)$.

To get the joint pdf, you can take derivatives like you could in the one dimensional case:

$$f(x_1, x_2, \cdots, x_k) = \frac{\partial^k}{\partial x_1 \partial x_2 \ldots \partial x_k} F(x_1, x_2, \ldots x_k)$$

(Use this when you are given the CDF)

$\boxed{\text{Thm } 4.3.1}$ Conditions of a joint pdf.

$f(x_1, x_2, \cdots, x_k) \geq 0$ for all x

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_k)\, dx_k \cdots dx_1 = 1$$

Example Let X_1 denote the concentration of a certain substance in one trial of an experiment, and X_2 the concentration of the substance on a second trial of an experiment.

Suppose $f(x_1, x_2) = 4x_1 x_2 I_{(0,1)}(x_1) I_{(0,1)}(x_2)$

The joint CDF is (for $0<x_1<1$ and $0<x_2<1$)

$$F(x_1, x_2) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(t_1, t_2)\, dt_1\, dt_2$$

$$= \int_0^{x_2} \int_0^{x_1} 4t_1 t_2 \, dt_1\, dt_2$$

$$= x_1^2 x_2^2, \quad 0<x_1<1, \ 0<x_2<1$$

If $X_1 > 1$ and $0 < X_2 < 1$, then

$$F(X_1, X_2) = X_2^2$$

If $X_2 > 1$ and $X_1 > 1$, then

$$F(X_1, X_2) = 1$$

If $X_2 > 1$ and $0 < X_1 < 1$, then

$$F(X_1, X_2) = X_1^2$$

$$\boxed{F(X_1, X_2) = \begin{cases} 1, & X_1 > 1, X_2 > 1 \\ X_1^2, & X_2 > 1, 0 < X_1 < 1 \\ X_2^2, & X_1 > 1, 0 < X_2 < 1 \\ X_1^2 X_2^2, & 0 < X_1 < 1, 0 < X_2 < 1 \end{cases}}$$

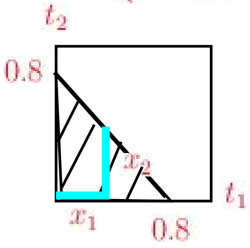
It is possible to evaluate joint prob by integrating over the appropriate region

$$P[X \in A] = \int_A \cdots \int f(x_1, x_2, \ldots, x_k)\, dx_1 \cdots dx_k$$

$\boxed{\text{EX:}}$ Suppose we want to find the prob. that the average concen. is less than 0.4

$$\text{Then } \frac{x_1 + x_2}{2} = 0.4 \Longrightarrow x_2 = -x_1 + 0.8$$

$$P\left(\frac{x_1 + x_2}{2} < 0.4\right) = \int_0^{0.8} \int_0^{-x_2+0.8} 4 t_1 t_2 \, dt_1 \, dt_2$$



$$= \frac{128}{1875} \text{ (verify it!!)}$$

DEF 4.3.2

If the pair (X_1, X_2) of cont R.V. has the joint pdf $f(x_1, x_2)$, then the marginal pdf's of X_1 & X_2 are

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

DEF 4.3.3.

$X = (X_1, \cdots, X_k)$ k dimensional R.V.

$F(x_1, \cdots, x_k)$ joint CDF. Then the marginal CDF is

$$F_j(x_j) = \lim_{\substack{x_i \to \infty \\ i \neq j}} F(x_1, \cdots, x_k)$$

The marginal pdf is

$$f_j(x_j) = \begin{cases} \displaystyle\sum \cdots \sum_{i \neq j} f(x_1, \cdots, x_j, \cdots, x_k) \\ \displaystyle\int \cdots \int_{i \neq j} f(x_1, \cdots, x_j, \cdots, x_k)\, d \cdots d \end{cases}$$

Ex. Let X_1, X_2, and X_3 be cont w/ joint pdf the form

$$f(x_1,x_2,x_3) = \begin{cases} c, & 0<x_1<x_2<x_3<1 \\ 0, & \text{otherwise} \end{cases}$$

Note. $c=6$ why? How?

Suppose we want the marginal of X_3

From $f_j(x_j)$ above,

$$f_3(x_3) = \int_0^{x_3}\int_0^{x_2} 6\,dx_1\,dx_2 \overset{\text{integration magic}}{=} \begin{cases} 3x_3^2, & \text{if } 0< x_3<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the joint pdf of $X_1 \& X_2$

$$f(x_1,x_2) = \int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_3$$

$$= \int_{x_2}^{1} 6\,dx_3 = 6x_3\Big|_{x_2}^{1}$$

$$= 6(1-x_2)$$

$$f(x_1,x_2) = \begin{cases} 6(1-x_2), & 0<x_1<x_2<1 \\ 0, & \text{otherwise} \end{cases}$$