[5.2] Prop. of Expected Values

Thm. If $X = (X_1, \cdots, X_k)$ has joint pdf $f(x_1, x_2, \cdots, x_k)$ and if $Y = U(X_1, X_2, \ldots, X_k)$ is a function of X, then

$$E(Y) = E_X[U(X_1, X_2, \ldots X_k)], \text{ where}$$

$$E_X[U(X_1, X_2, \ldots, X_k)] =$$

$$\begin{cases} \sum_{x_1} \cdots \sum_{x_k} U(x_1, \ldots, x_k) f(x_1, \ldots x_k), & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} U(x_1, \ldots, x_k) f(x_1, \ldots x_k) dx_1 dx_2 \ldots dx_k, & \text{if } x \text{ is continuous} \end{cases}$$

Thm 5.2.2

If X_1 & X_2 are R.V with joint pdf $f(x_1, x_2)$,

then $E(X_1 + X_2) = E(X_1) + E(X_2)$

proof : (discrete case)

swap x1 and x2

$$E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2) f(x_1, x_2)$$

$$= \sum_{x_1} \sum_{x_2} x_1 f(x_1, x_2) + \sum_{x_1} \sum_{x_2} x_2 f(x_1, x_2)$$

$$= \sum_{x_1} x_1 \underbrace{\sum_{x_2} f(x_1, x_2)}_{f_1(x_1)} + \sum_{x_2} x_2 \underbrace{\sum_{x_1} f(x_1, x_2)}_{f_2(x_2)}$$

$$= \sum_{x_1} x_1 f_1(x_1) + \sum_{x_2} x_2 f_2(x_2)$$

$$= E(X_1) + E(X_2)$$

$\boxed{\text{Thm}}$ If X & Y are indep R.V.

and $g(x)$ & $h(y)$ are functions, then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

proof. cont. case is in the book.

DEF. The covariance of a pair of random variables X & Y is defined by

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sigma_{XY}$$

Thm. If X & Y are RV, and a, b are const then

$$\boxed{\begin{aligned} &a)\ Cov(aX, bY) = ab\,Cov(X,Y) \\ &b)\ Cov(X+a, Y+b) = Cov(X,Y) \\ &c)\ Cov(X, aX+b) = a\,Var(X) \end{aligned}}$$

Thm 5.2.5 If X & Y are RV's, then

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$$ and

If X & Y are independent, then

$$\boxed{Cov(X,Y) = 0.}$$

Proof: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$

$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y$

$= E(XY) - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$

$= E(XY) - \mu_X\mu_Y$

If X, Y are indep, then $Cov(X,Y) = E(X)E(Y) - \mu_X\mu_Y$
$= \mu_X\mu_Y - \mu_X\mu_Y = 0$

$\boxed{\text{Thm } 5.26}$

If X_1 & X_2 are RV w/pdf $f(x_1, x_2)$,

$$V(X_1+X_2) = V(X_1) + V(X_2) + 2Cov(X_1, X_2)$$

If $X_1 \perp\!\!\!\perp X_2$ ($\perp\!\!\!\perp$ = is independent of), then

$$V(X_1+X_2) = V(X_1) + V(X_2)$$

proof: DEF: $\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$

$$\begin{aligned}
V(X_1+X_2) &= E(X_1+X_2)^2 - \left[E(X_1+X_2)\right]^2 \\
&= E\left[X_1^2+2X_1X_2+X_2^2\right] - \Big[\underbrace{E(X_1)}_{\mu_1}+\underbrace{E(X_2)}_{\mu_2}\Big]^2 \\
&= E(X_1^2) + 2E(X_1X_2) + E(X_2^2) - \mu_1^2 - 2\mu_1\mu_2 - \mu_2^2 \\
&= \left[E(X_1^2)-\mu_1^2\right] + \left[E(X_2^2)-\mu_2^2\right] + 2\left[E(X_1X_2)-\mu_1\mu_2\right] \\
&= V(X_1) + V(X_2) + 2Cov(X_1,X_2)
\end{aligned}$$

Note: If $X_1 \perp\!\!\!\perp X_2$, then

$$V(X_1+X_2) = V(X_1) + V(X_2)$$

In general,

(a) $E\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i E(X_i)$

(b) $V\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i^2 V(X_i) + 2\sum_{i<j}\sum a_i a_j Cov(X_i, X_j)$

(c) If $X_1, X_2, \ldots, X_k$ are indep, then

$$V\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i^2 V(X_i)$$

Ex: Suppose $Y \sim BIN(n, p)$

Because binomial R.V.'s result from n independent Bernoulli R.V.'s, then

$$Y = \sum_{i=1}^{n} X_i, \quad \text{where } X_i \sim BER(p)$$

$$\boxed{\begin{aligned} E[Y] &= E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np \\ V[Y] &= V\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} pq = npq \end{aligned}}$$

application of linearity of E() operator (part a)

application of linearity of V() operator (part c)
(only under independence!)