[5.5] Joint Moment Generating Functions

Def: The joint MGF of $Y = (X_1, \ldots, X_k)$, if it exists, is defined as:

$$M_X(t) = M_{X_1,\ldots,X_k}(t_1, \ldots, t_k) = E\left[\exp\left(\sum_{i=1}^{k} t_i X_i\right)\right]$$

where $t = (t_1, \ldots, t_k)$ and $0 < h < |t_i|$

Thm 5.5.1 If $M_{X,Y}(t_1, t_2)$ exists, then $X \perp\!\!\!\perp Y$ (X and Y are independent) iff

$$M_{X,Y}(t_1, t_2) = M_X(t_1) M_Y(t_2)$$

Mixed Moments

$$\mathrm{E}[X_i^r X_j^s] = \frac{\partial^r}{\partial t_i^r} \frac{\partial^s}{\partial t_j^s} M_X(t_1, \ldots, t_k)\Big|_{t_i=0}$$

Marginal dist MGF's

$$M_X(t_1) = M_{X,Y}(t_1, 0)$$
$$M_Y(t_2) = M_{X,Y}(0, t_2)$$

EX: $X = (X_1, \ldots, X_k) \sim \text{MULT}(n, p_1, \ldots, p_k)$ where $p_{k+1} = 1 - \Sigma\, p_i$, $x_{k+1} = n - \Sigma\, x_i$

Marginals $\Longrightarrow X_i \sim \text{BIN}(n, p_i)$

The Joint MGF is: $M_X(t) = \text{E}[\exp(\Sigma\, t_i X_i)]$

$$= \sum \cdots \sum \frac{n!}{x_1! \cdots x_k! x_{k+1}!} (p_1 e_1^t)^{x_1} \cdots (p_k e^{t_k})^{x_k} p_{k+1}^{x_{k+1}}$$

$$= (p_1 e^{t_1} + \cdots + p_k e^{t_k} + p_{k+1})^n$$

Suppose $k = 3$, then

Thus, $M_{X_1, X_2, X_3}(t_1, t_2, t_3) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3 e^{t_3} + p_4)^n$,

where $p_4 = 1 - p_1 - p_2 - p_3$

$$M_{X_1, X_2}(t_1, t_2) = M_{X_1, X_2, X_3}(t_1, t_2, 0) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3 e^0 + 1 - p_1 - p_2 - p_3)^n,$$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + p_3 + 1 - p_1 - p_2 - p_3)^n,$$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + 1 - p_1 - p_2)^n,$$

By inspection, we recognize the MGF as: $(X_1, X_2) \sim \text{MULT}(n, p_1, p_2)$

Ex: Suppose $(X, Y) \sim \mathrm{BVN}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

We can find the joint MGF for this by evaluating the integral:

$$M_{X,Y}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(t_1 x + t_2 y) f(x, y)\,dx\,dy$$

It's a bit difficult, so let's use some theorems:

We use both $M_X(t) = \exp\{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2\}$ and $M_{Y|X}(t) = \exp\left\{\mu_2 t + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1)t + \frac{1}{2}\sigma_2^2(1-\rho^2)t^2\right\}$

$$M_{X,Y}(t_1, t_2) = \mathrm{E}(\exp\{t_1 X + t_2 Y\}) = \mathrm{E}_X[\mathrm{E}(\exp\{t_1 X + t_2 Y\}|X)] = \mathrm{E}_X[\exp\{t_1 X\}\mathrm{E}(\exp\{t_2 Y\}|X)]$$

$$= \mathrm{E}_X[\exp\{t_1 X\} M_{Y|X}(t_2)] \quad \text{where } M_{Y|X}(t) = \exp\left\{\mu_2 t + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1)t + \frac{1}{2}\sigma_2^2(1-\rho^2)t^2\right\}$$

Collect the parts that don't depend on X. Then pull them out of the expectation (call them "c")

$$= c \cdot E_X\left[\exp\left\{\left(t_1 + \rho\frac{\sigma_2}{\sigma_1}t_2\right)X\right\}\right] = c \cdot M_X\left(t_1 + \rho\frac{\sigma_2}{\sigma_1}t_2\right)$$

$$= c \cdot \exp\left\{\mu_1\left(t_1 + \rho\frac{\sigma_2}{\sigma_1}t_2\right) + \frac{1}{2}\sigma_1^2\left(t_1 + \rho\frac{\sigma_2}{\sigma_1}t_2\right)^2\right\}$$

For more details, the complete derivation is on the class website. Look for "Joint MGFs - Bivariate Normal Distribution".

Substitute c back in and simplify the expression. You will find it is (after lotsa algebra)

$$= \exp\left\{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + \rho\sigma_1\sigma_2 t_1 t_2\right\}$$