# [6.3] Joint Transformations

Thm 6.3.5

If $X$ is a vector of discrete R.V. with joint pdf $f_X(x)$ and $Y = u(X)$ defines a one to one transformation, then joint pdf of $Y$ is

$$f_Y(y) = f_X(x) \text{ , where}$$

$$y = (y_1, y_2, \cdots, y_k), x = (x_1, x_2, \cdots, x_k)$$

and $x$ is the solution of the transformation

$$y = u(x). \quad (x \text{ depends on } y.)$$

If it is not 1-1, split it up over intervals where it is. $A_j$

Then the equation $y = u(x)$ has a unique sol'n $x = x_j$ or $x_j = (x_{1j}, x_{2j}, \ldots x_{kj})$ over $A_j$. Then the pdf is

$$f_Y(y) = \sum_j f_X(x_j)$$

Note that for discrete random variables, THERE IS NO JACOBIAN

joint transforms of continuous R.V.'s can be accomplished, but the Jacobian has to be generalized. Suppose $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ are functions and $x_1$ & $x_2$ are unique solutions to $y_1 = u_1(x_1, x_2)$, $y_2 = u_2(x_1, x_2)$.

Then the Jacobian of the transformation is:

$$J = \begin{vmatrix} \dfrac{\partial X_1}{\partial Y_1} & \dfrac{\partial X_1}{\partial Y_2} \\ \dfrac{\partial X_2}{\partial Y_1} & \dfrac{\partial X_2}{\partial Y_2} \end{vmatrix}$$

Ex: Suppose we want to transform $X_1$ & $X_2$ into $X_1$ & $X_1 X_2$. Then

$$y_1 = x_1 \implies x_1 = y_1$$
$$y_2 = x_1 x_2 \implies x_2 = \frac{y_2}{x_1} = \frac{y_2}{y_1}$$

Note:

$$Y = U(\underbrace{X_1, X_2}_{X}), \text{ where } U(x_1, x_2) = (x_1, x_1 x_2)$$

$$X = U^{-1}(Y) = U^{-1}(Y_1, Y_2) = (Y_1, Y_2/Y_1)$$

So
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{vmatrix} = \frac{1}{y_1}$$

For a general transform $y = u(x)$, that has a unique solution $x = (x_1, \ldots, x_k)$

the Jacobian is the determinant of $k \times k$ matrix:

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \frac{\partial x_k}{\partial y_2} & \cdots & \frac{\partial x_k}{\partial y_k} \end{vmatrix}$$

EX:

$X_1 \sim EXP(1)$ $\quad X_1 \perp\!\!\!\perp X_2$

$X_2 \sim EXP(1)$

$$f(x_1, x_2) = e^{-(x_1+x_2)}$$

Transform from $(x_1, x_2)$ to $(x_1 - x_2, x_1 + x_2)$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} = u(x) = u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{array}{r} y_1 = x_1 - x_2 \\ y_2 = x_1 + x_2 \\ \hline y_1 + y_2 = 2x_1 \end{array} \qquad \begin{array}{r} -y_1 = -x_1 + x_2 \\ y_2 = x_1 + x_2 \\ \hline y_2 - y_1 = 2x_2 \end{array}$$

$$x_1 = \frac{y_1 + y_2}{2} \qquad x_2 = \frac{y_2 - y_1}{2}$$

So $x = u^{-1}(Y) = \begin{pmatrix} \frac{Y_1 + Y_2}{2} \\ \frac{Y_2 - Y_1}{2} \end{pmatrix}$

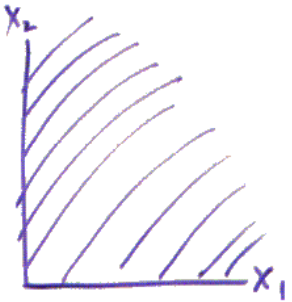
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$f_Y(y) = f_X(x)\,|J|$$

$$= f_X\left(\frac{y_1+y_2}{2}, \frac{y_2-y_1}{2}\right) \cdot \frac{1}{2}$$

$$= e^{-\left(\frac{y_1+y_2}{2} + \frac{y_2-y_1}{2}\right)} \cdot \frac{1}{2}$$

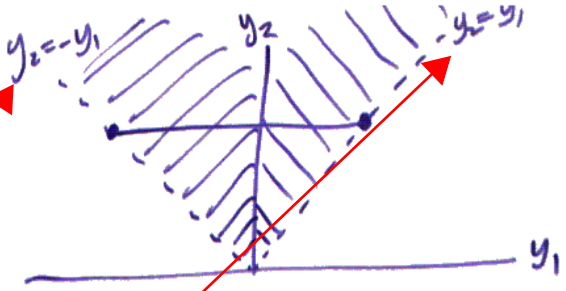
$$= \frac{1}{2} e^{-y_2}, \quad y \in B \Rightarrow \text{what is } B?$$




$A = \{(x_1, x_2) \mid f(\underset{\uparrow}{x}) > 0\}$

$x_1, x_2$

$x_1 > 0,\ x_2 > 0.$




The transform $x_1 = \dfrac{y_1 + y_2}{2} > 0$

$2x_1 = y_1 + y_2 > 0 \Rightarrow y_2 > -y_1$

$x_2 = \dfrac{y_2 - y_1}{2} > 0$

$\Rightarrow 2x_2 = y_2 - y_1 > 0 \Rightarrow y_2 > y_1$

So $y_2 > 0$ and $-y_2 < y_1 < y_2$

$|y_1| < y_2 < \infty$

$f_Y(y) = \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2)\, I_{(|y_1|,\infty)}(y$

The marginal for $Y_1$ is:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} I_{(|y_1|,\infty)}(y_2)\, I_{(0,\infty)}(y_2)\,dy_2$$

$$= \int_{|y_1|}^{\infty} \frac{1}{2} e^{-y_2}\,dy_2 = -\frac{1}{2} e^{-y_2}\Big|_{|y_1|}^{\infty}$$

$$= \frac{1}{2} e^{-|y_1|}$$

$$Y_1 \sim DE(1,0)$$

The marginal for $Y_2$ is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_Y(y)\,dy_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} \underbrace{I_{(|y_1|,\infty)}(y_2)}_{\downarrow}\, I_{(0,\infty)}(y_2)\,dy_1$$

$$|y_1| < y_2 \Rightarrow \underbrace{-y_2 < y_1 < y_2}_{I_{(-y_2,y_2)}(y_1)}$$

$$= \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2)\,dy_1$$

$$= \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2) \int_{-y_2}^{y_2} dy_1$$

$$= \frac{1}{8} e^{-y_2} I_{(0,\infty)}(y_2)\; 8 y_2$$

$$= y_2 e^{-y_2} I_{(0,\infty)}(y_2) \Longrightarrow Y_2 \sim GAM(1,2)$$

Ex: Let $X_i \sim GAM(\theta, K_i)$ $i=1,2$

$X_1 \perp\!\!\!\perp X_2$

We want to find the dist of

$$\frac{X_1}{X_1+X_2} \quad \text{and} \quad \underbrace{\frac{X_2}{X_1+X_2}}_{\text{do later.}}$$

$$Z_1 = \frac{X_1}{X_1+X_2} \qquad Z_2 = X_2$$

$$X_1 = \frac{Z_1 Z_2}{1-Z_1} \qquad X_2 = Z_2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} = \begin{vmatrix} \frac{z_2}{(1-z_1)^2} & \frac{z_1}{1-z_1} \\ 0 & 1 \end{vmatrix} = \frac{z_2}{(1-z_1)^2}$$

The pdf of $X_i$ is

$$f_{X_i}(x_i) = \frac{1}{\Gamma(K_i)\theta^{K_i}} x_i^{K_i-1} e^{-\frac{x_i}{\theta}} I_{(0,\infty)}(x_i)$$

The joint pdf of $X_1$ & $X_2$ is

$$f_{X_1,X_2}(x_1,x_2) = f_1(x_1) \cdot f_2(x_2)$$

$$= \frac{1}{\Gamma(K_1)\Gamma(K_2)\,\theta^{K_1+K_2}} x_1^{K_1-1} x_2^{K_2-1} e^{-\frac{x_1+x_2}{\theta}} I_{(0,\infty)}(x_1) \times I_{(0,\infty)}(x_2)$$

So

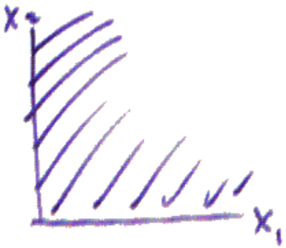
$$f_z(z) = f_x(x)\,|J|$$

$$= f_x\left(\frac{z_1 z_2}{1-z_1},\ z_2\right)\frac{z_2}{(1-z_1)^2}$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}}\left(\frac{z_1 z_2}{1-z_1}\right)^{k_1-1} z_2^{k_2-1}\, e^{-\frac{1}{\theta}\left(\frac{z_1 z_2}{1-z_1}+z_2\right)}\frac{z_2}{(1-z_1)^2}\, I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right) I_{(0,\infty)}(z_2)$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}}\,\frac{z_1^{k_1-1}\, z_2^{k_1+k_2-1}}{(1-z_1)^{k_1+1}}\, e^{-\frac{1}{\theta}\underbrace{\left(\frac{\cancel{z_1 z_2}+z_2(1-\cancel{z_1})}{1-z_1}\right)}_{\frac{z_2}{1-z_1}}}\, I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right) I_{(0,\infty)}(z_2)$$

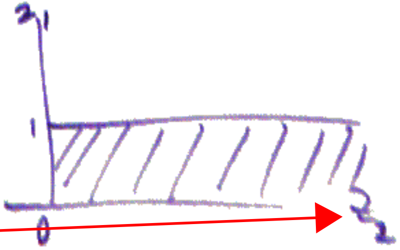
$$\to I_{(0,1)}(z_1)$$




Note that $\underbrace{0 < \frac{z_1 z_2}{1-z_1} < \infty}$

$0 < z_1 < 1 \qquad 0 < z_2 < \infty$

$\Rightarrow$



We want the dist of $Z_1$

$$f(z_1) = \int_{-\infty}^{\infty} f_z(z)\, dz_2$$

$$= \underbrace{\frac{I_{(0,1)}(z_1)\, z_1^{k_1-1}}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}(1-z_1)^{k_1+1}}}_{c(z_1)} \int_0^\infty z_2^{k_1+k_2-1}\, e^{-\frac{z_2}{\theta(1-z_1)}}\, dz_2$$

$$u = \frac{z_2}{\theta(1-z_1)} \Rightarrow z_2 = u\theta(1-z_1)$$

$$dz_2 = \theta(1-z_1)\, du$$

$$B(k_1,k_2) = \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \int_0^1 z_1^{k_1-1}(1-z_1)^{k_2-1}\, dz_1$$

Try and prove it! ~~If~~ If you do, I'll add 5% to any one of your tests.

$$= c(z_1)\int_0^\infty \left[u\theta(1-z_1)\right]^{k_1+k_2-1} e^{-u}\, \theta(1-z_1)\, du$$

$$= c(z_1)\, \theta^{k_1+k_2}(1-z_1)^{k_1+k_2} \underbrace{\int_0^\infty u^{k_1+k_2-1} e^{-u}\, du}_{\Gamma(k_1+k_2)} = \frac{I_{(0,1)}(z_1)\, z_1^{k_1-1}\, \Gamma(k_1+k_2)\, \cancel{\theta^{k_1+k_2}}\, (1-z_1)^{\cancel{k_1}+k_2}}{\Gamma(k_1)\Gamma(k_2)\, \cancel{\theta^{k_1+k_2}}\, (1-z_1)^{\cancel{k_1}+1}}$$

$$= \frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}\, z_1^{k_1-1}(1-z_1)^{k_2-1}\, I_{(0,1)}(z_1)$$

$Z_1 \sim BETA(k_1,k_2)$

## Thm 6.3.3 Probability Integral Transformation

If $X$ is continuous with CDF $F(x)$, then $U = F(X) \sim UNIF(0,1)$

Proof: Special case (where $F^{-1}$ exists)

$$F_U(u) = P[U \le u] = P[F(X) \le u]$$

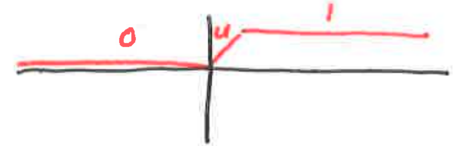
$$= P[X \le F^{-1}(u)]$$

$$= F(F^{-1}(u)) = u$$

Because $0 \le F(X) = u \le 1$

then $F_U(u) = \begin{cases} 0, & u \le 0 \\ u, & 0 < u < 1 \\ 1, & u \ge 1 \end{cases}$

It follows that this is the CDF for the Uniform(0,1) dist




The pdf is

$$f_u(u) = \begin{cases} 0, & u \le 0 \\ 1, & 0 < u < 1 \\ 0, & u \ge 1 \end{cases}$$

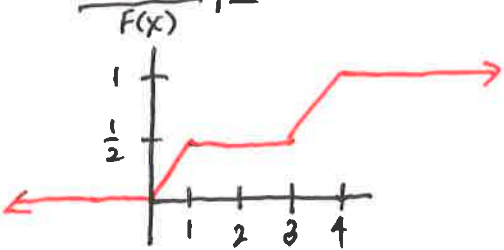
$$= \begin{cases} 1, & 0 < u < 1 \\ 0, & \text{otherwise} \end{cases}$$

A more generalized choice for $F^{-1}$ is

$$G(u) = \min\{x \mid u \le F(x)\}$$

This exists for any CDF, and agrees with $F^{-1}$ if $F$ is 1-1. The only parts that are diff are where $F$ is constant

## Example




$$F(u) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}x, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 3 \\ \frac{1}{2}(x-2), & 3 < x \leq 4 \\ 1, & x \geq 4 \end{cases}$$

Note: $G(\frac{1}{4}) \Rightarrow \frac{1}{2}x = \frac{1}{4} \Rightarrow x = 1/2 \Rightarrow G(\frac{1}{4}) = \frac{1}{2}$

$G(1/2) \Rightarrow \min(F(x) = 1/2) = 1 \Rightarrow G(\frac{1}{2}) = 1$

$G(1) \Rightarrow \min(F(x) = 1) = 4$

So

$$G(u) = \begin{cases} 2u, & 0 \leq u \leq 1/2 \\ 2(u+1), & \frac{1}{2} < u \leq 1 \end{cases}$$

Next,

$\boxed{\text{Thm } 6.3.4}$

Let $F(x)$ be a CDF and let $G(u)$ be defined as

$$G(u) = \min\{x \mid u \leq F(x)\}$$

Then

If $U \sim UNIF(0,1)$, then

$$X = G(U) \sim F(x)$$

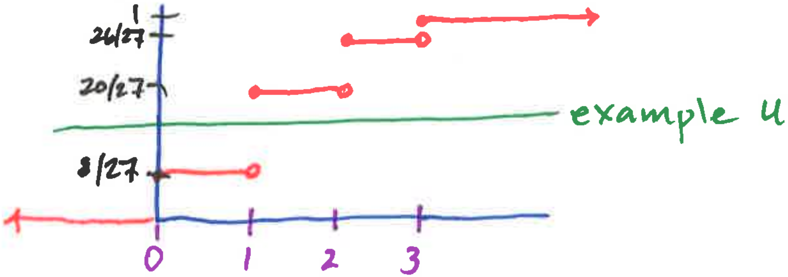
Note: the thm does not require $F(x)$ to be continuous!

EX: $X \sim BIN(3, 1/3)$

Then

| $x$ | $f(x)$ | $F(x)$ |
|---|---|---|
| 0 | $8/27$ | $8/27$ |
| 1 | $4/9$ | $20/27$ |
| 2 | $2/9$ | $26/27$ |
| 3 | $1/27$ | 1 |

Graph:




Green line $\Rightarrow \frac{8}{27} < u \le \frac{20}{27} \Rightarrow G(u) = 1$

Thus,

$$G(u) = \begin{cases} 0, & 0 < u \le \frac{8}{27} \\ 1, & \frac{8}{27} < u \le \frac{20}{27} \\ 2, & \frac{20}{27} < u \le \frac{26}{27} \\ 3, & \frac{26}{27} < u \le 1 \end{cases}$$

This means $G(u) \sim BIN(3, 1/3)$

So we can generate random samples of Binomial random numbers. First, generate random numbers from $Unif(0,1)$

$$\underbrace{\begin{array}{l} u_1 = .01609 \\ u_2 = .37749 \\ u_3 = .22523 \\ u_4 = .74125 \end{array}}_{\text{Random sample from UNIF}(0,1)} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \underbrace{\begin{array}{l} X_1 = 0 \\ X_2 = 1 \\ X_3 = 0 \\ X_4 = 2 \end{array}}_{\text{Random sample from } BIN(3, \frac{1}{3})}$$