[6.4] Sums of Random Variables

Convolution formula

$S = X_1 + X_2 \Rightarrow f(x_1, x_2)$ joint pdf.

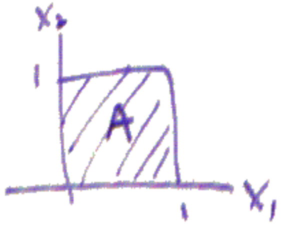
$$f_S(s) = \int_{-\infty}^{\infty} f(t, s-t)\, dt$$

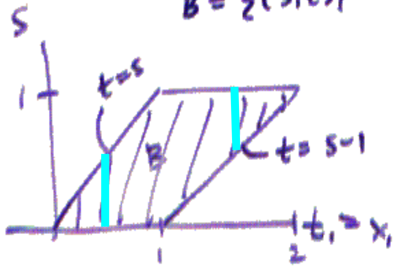
Ex: Let $X_i \sim UNIF(0,1)$

$X_1 \perp\!\!\!\perp X_2$

Let $S = X_1 + X_2$

$B = \{(s,t) \mid 0 < t < s < t+1 < 2\}$





$$f_S(s) = \int_0^s dt = s, \quad 0 < s < 1$$

$$= \int_{s-1}^{1} dt = 2 - s, \quad 1 < s < 2$$

$$= 0, \quad \text{otherwise.}$$

Moment generating function method

Thm: If $X_1, \dots, X_n$ are independent R.V. with MGF $M_{X_i}(t)$, then MGF of $Y = \sum_{i=1}^{n} X_i$ is

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Proof: (Easy!)

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t\Sigma X_i}\right]$$

$$= E\left[e^{tX_1 + tX_2 + tX_3 \cdots + tX_n}\right]$$

$$= E\left[e^{tX_1} e^{tX_2} e^{tX_3} \cdots e^{tX_n}\right]$$

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_n}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{tX_i}\right] = \prod_{i=1}^{n} M_{X_i}(t)$$

Note that if $X_i \sim iid$, then

iid = independent and identically distributed

$$M_Y(t) = \left[M_{X_1}(t)\right]^n$$

Ex: $X_i \sim BIN(n_i, p)$ X_i is indep

let $Y = \sum_{i=1}^{n} X_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (pe^t + q)^{n_i} = (pe^t + q)^{\Sigma n_i}$$

$Y \sim \mathrm{BIN}(\Sigma n_i, p)$

Ex: $X_i \sim POI(\mu_i)$ X_i is indep.

let $Y = \sum_{i=1}^{n} X_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} exp\left[\mu_i (e^t - 1)\right] = exp\left[\Sigma \mu_i (e^t - 1)\right]$$

$Y \sim \mathrm{POI}(\Sigma \mu_i)$

EX: $X_i \sim GAM(\beta, \alpha_i),$ where $X_i \sim id$

Let $Y = \sum_{i=1}^{n} X_i$

$$M_y(t) = E(e^{tY}) = E(e^{t\sum x_i})$$

$$= E\left(e^{tX_1} e^{tX_2} \cdots e^{tX_n}\right)$$

$$= E\left(e^{tX_1}\right) E\left(e^{tX_2}\right) \cdots E\left(e^{tX_n}\right)$$

$$= M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{1-\beta t}\right)^{\alpha_i}$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \left(\frac{1}{1-\beta t}\right)^{\alpha_2} \cdots \left(\frac{1}{1-\beta t}\right)^{\alpha_n}$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_n}$$

$$= \left(\frac{1}{1-\beta t}\right)^{\sum \alpha_i}$$

So, it follows $Y \sim GAM(\beta, \sum \alpha_i)$