[7.2] Sequences of Random Variables.

Consider a sequence of Random vars $Y_1, Y_2, \ldots$
with corresponding CDFs $G_1(y), G_2(y), \ldots$
where

$$G_n(y) = P[Y_n \le y]$$

<u>DEF</u>: If $Y_n \sim G_n(y)$ for $n \in \mathbb{Z}^+$,
and if some CDF $G(y)$ such that

$$\lim_{n\to\infty} G_n(y) = G(y)$$

for all values y at which $G(y)$ is continuous,
then the sequence $Y_1, Y_2, \ldots$ is
said to <u>convergeindistribution</u> to $Y \sim G(y)$
or also denoted as

$$Y_n \xrightarrow{d} Y.$$

The distribution corresponding to $G(y)$ is called the <u>Limiting Distribution</u> of Y_n

Ex: Let $X_1, X_2, \cdots X_n$ be a RS from UNIF(0,1) ($X_i \sim UNIF(0,1)$)

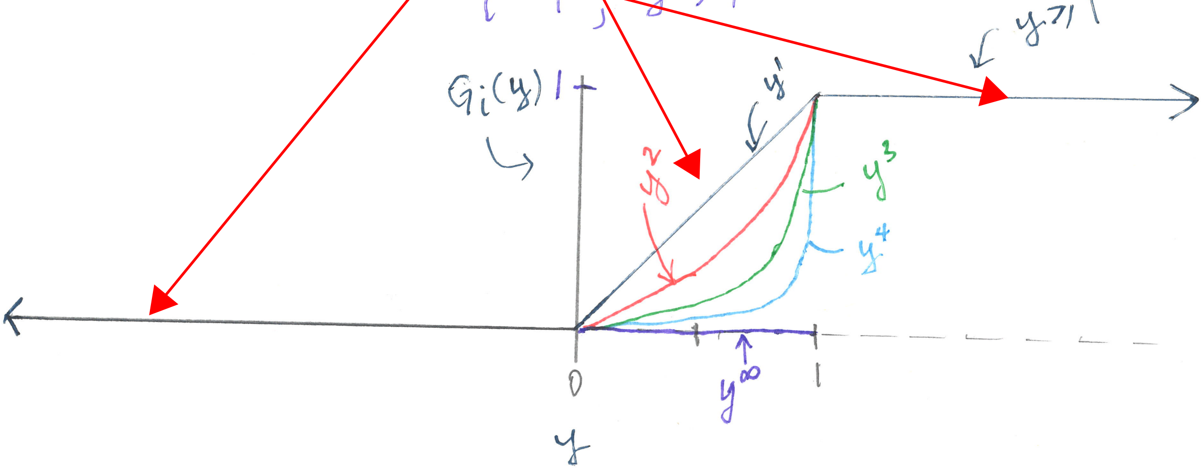
The distribution of largest order stat is $G_n(y) = [F(y)]^n$

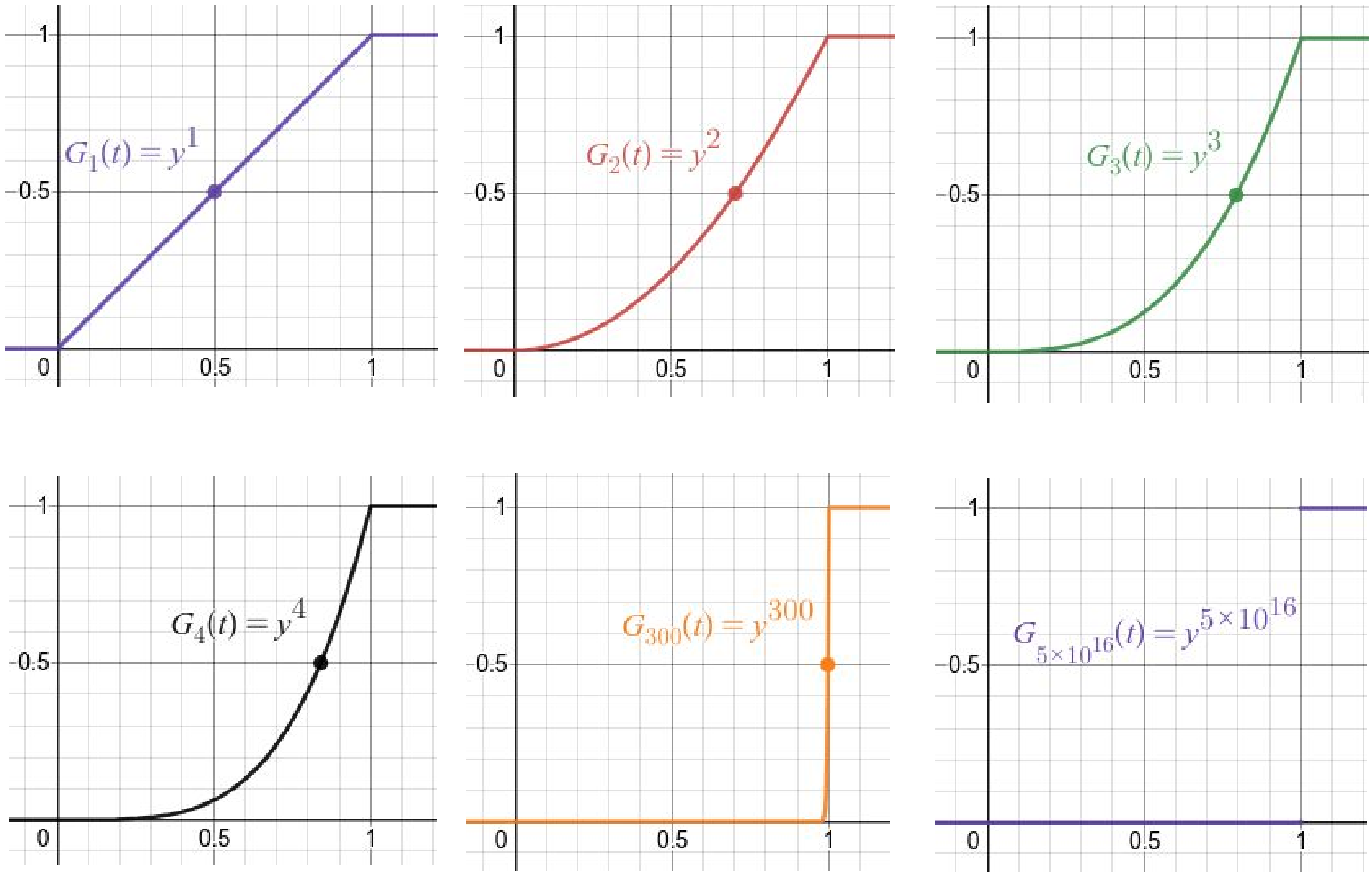
Since the uniform dist has the CDF $F(y) = \begin{cases} 0, & y \le 0 \\ y, & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$

then

$$G_n(y) = \begin{cases} 0, & y \le 0 \\ y^n, & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$$

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$





Desmos finally displayed the discontinuity when n was large enough.
Note, however, the discontinuity exists ONLY IN THE LIMIT.

Note that in the limit,

$$\lim_{n\to\infty} G_n(y) = \begin{cases} \lim_{n\to\infty} 0, & y<0 \\ \lim_{n\to\infty} y^n, & 0<y<1 \\ \lim_{n\to\infty} 1, & y\geq 1 \end{cases} = \begin{cases} 0, & y<0 \\ 0, & 0<y<1 \\ 1, & y\geq 1 \end{cases} = G(y)$$

$$G(y) = \begin{cases} 0, & y<1 \\ 1, & y\geq 1 \end{cases}$$

This distribution has been encountered before! This is called the Degenerate distribution, a discrete dist that concentrates all its probability on one spot.

The CDF of a DEG(c) dist is $G(y) = \begin{cases} 0, & y<c \\ 1, & y\geq c \end{cases}$

The pdf is $g(y) = \begin{cases} 0, & y\neq c \\ 1, & y=c \end{cases}$

The MGF is $M(t) = e^{ct}$

Why find limiting Distributions?

It allows us to generalize what happens with a large sample size. We can then use the <u>limiting</u> dist as an approximation to the actual dist.

This is why the Normal dist is so often used! It appears as a limiting distribution a lot!

In section 7.3, we will learn of the most famous thm. in all statistics! (Central Limit theorem)

Let me give you an example:

Let X = the time until a component fails

So $X \sim EXP(\theta)$ and $\underbrace{f(x) = \frac{1}{\theta} e^{-x/\theta}}_{\text{pdf}}$ and $\underbrace{F(x) = 1 - \exp(-x/\theta)}_{\text{CDF}}, x > 0$

Let's take a random sample of n components.

We would like to model the shortest time to failure

So, that will be the smallest order stat $W = X_{1:n}$

We know that the CDF of W is

$$G_n(w) = 1 - [1 - F(w)]^n = 1 - [1 - (1 - \exp(-w/\theta))]^n, w > 0$$

$$= 1 - \left(e^{-\frac{w}{\theta}}\right)^n = 1 - e^{-\frac{nw}{\theta}}, w > 0$$

Remember that the CDF can be written piecewise as

$$G_n(w) = \begin{cases} 0, & w \leq 0 \\ 1 - e^{-\frac{nw}{\theta}}, & w > 0 \end{cases}$$

So $W \xrightarrow{d} DEG(0)$

or $W \xrightarrow{p} 0$

It follows that $G(w) = \lim_{n \to \infty} G_n(w) = \begin{cases} 0, & w \leq 0 \\ 1, & w > 0 \end{cases}$

Thus, the shortest time to failure is 0! It converges to the degenerate dist at $w=0$

Similarly, let's model the longest time to failure

We'll use the largest order stat $Z = X_{n:n}$

We know that the CDF of Z is

$$G_n(z) = [F(z)]^n = \left[1 - e^{-\frac{z}{\theta}}\right]^n, \; z>0$$

Since $\lim_{n\to\infty} G_n(z) = \lim_{n\to\infty} \left[1 - e^{-z/\theta}\right], (z>0) = 0$, then

$$G(y) = \lim_{n\to\infty} G_n(z) = \begin{cases} 0, & z<0 \\ 0, & z>0 \end{cases} = 0$$

So $G(y)$ is always zero. There is no limiting dist.

However, we can modify z and get something that does.

Let's define $Y = \frac{1}{\theta} Z - \ln(n)$.

Inverting we get $Z = \theta(Y + \ln(n))$

Note the support set is now $y > -\ln(n)$

The CDF of Y is

$$F_Y(y) = G_n(z) = [F(z)]^n = [1 - \exp(-z/\theta)]^n$$

$$= \left[1 - \exp\left(-\frac{\cancel{\theta}(y + \ln(n))}{\cancel{\theta}}\right)\right]^n = \left[1 - e^{-y} e^{-\ln n}\right]^n$$

$$= \left[1 - e^{-y} e^{\ln(n^{-1})}\right]^n = \left[1 - \frac{e^{-y}}{n}\right]^n$$

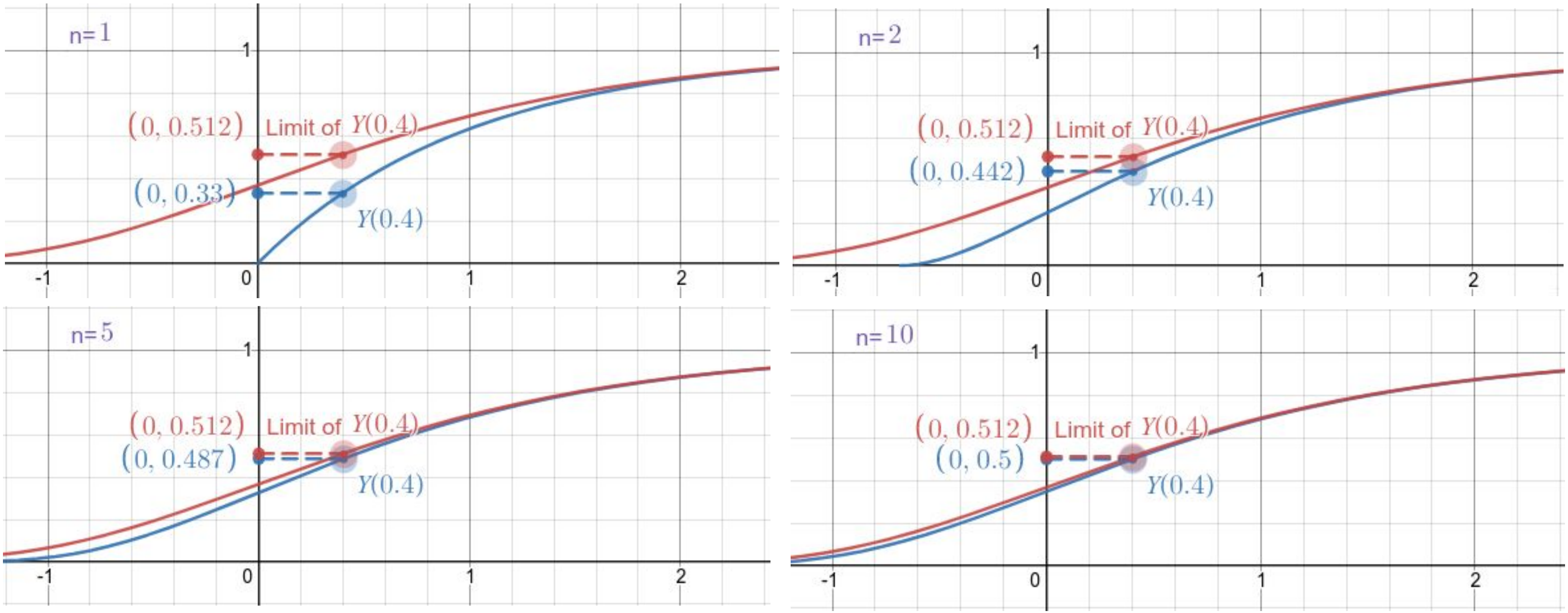
NOTE!!

So the limiting distribution of Y is $Y \xrightarrow{d} EV(1, 0)$

$$G = \lim_{n\to\infty} F_Y(y) = \lim_{n\to\infty}\left[1 - \frac{e^{-y}}{n}\right]^n = \exp(-e^{-y}), \quad \cancel{y > -\ln(n)}$$

no restrictions on y! $\Rightarrow$ no take limit too!

Watch the convergence as n increases.
Note that "Limit of Y(t)" is the limiting distribution and the "Y(t)" is G n(t) (original dist for current n).



Note how the support set for G_n stretches to the left (x>-ln(n)).
In the limit, what will the support set be?

What we are seeing is the convergence of G_n distribution to the limiting distribution as n goes to infinity. (Limiting dist is EV(1,0))

Example of how/what a limiting dist is compared to orig.

Suppose $T = X_{10:10}$ ($\theta=1$ for this), when we have a sample of size 10.

The CDF of T is $F_T(t) = (1-e^{-t})^{10}$

we learned that the largest order stat has no limiting dist, but that the related quantity $Y = T - \ln(n)$ does.

So $F_T(t) = P[T \le t] = P[Y + \ln(10) \le t] = P[Y \le t - \ln(10)]$

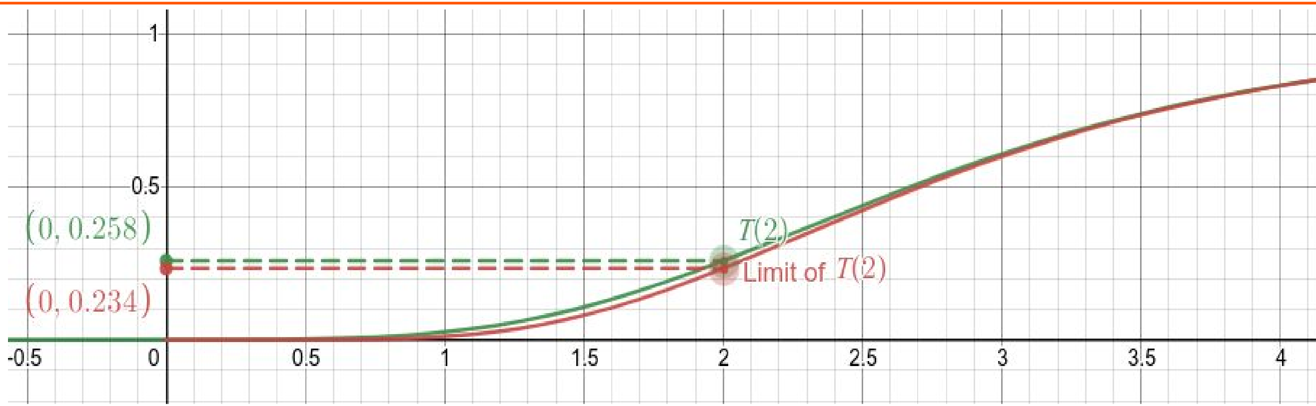
approx equal $\doteq G(t - \ln 10) = \exp\left(-e^{-(t-\ln 10)}\right)$

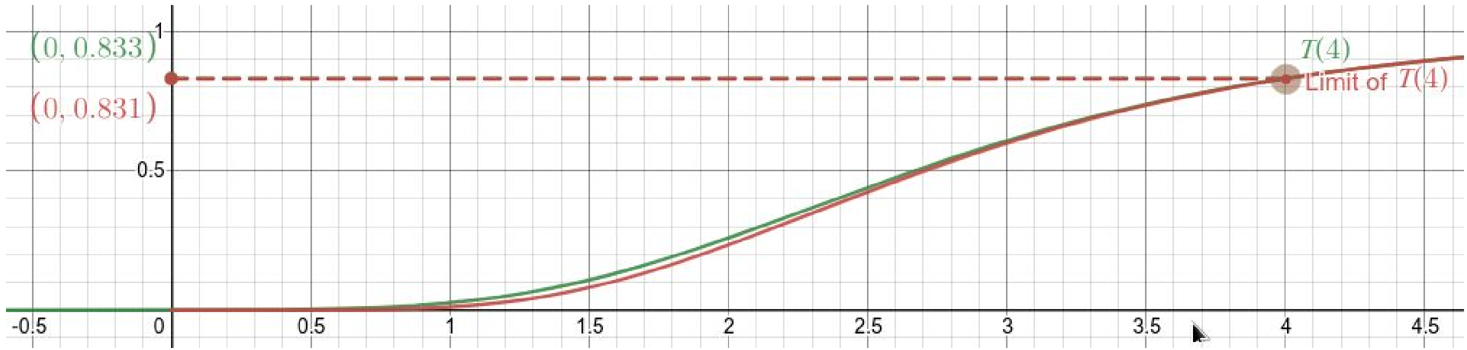
$= \exp(-10e^{-t})$ (the Limiting dist)

Compare it to $F_T(t)$ above!

So $F(t) = (1-e^{-t})^{10} \doteq \exp(-10e^{-t})$

This is the difference between the original CDF (G_n) and the limiting dist G(y) for the constant value of n=10. It is close to the previous slides on Y, but different in that n is contstant here. Rather, t is increasing.





The T(t) curve is the original CDF (1-e^(-t))^10 and "Limit of T(t)" is exp(-10e^(-t)).
As t increases, probabilities produced are very close. In practice, we will be using Limiting distributions when they give us an advantage. Note that as t increases, the difference between original vs limiting gets smaller and smaller.