[7.5] Asymptotic Normal Distributions.

DEF (Asymptotic Normal dist)

If $Y_1, Y_2, \dots$ is a sequence of RV's and m, c are constants such that

$$Z_n = \frac{Y_n - m}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

or $Y_n \dot\sim N(m, c^2/n)$

as $n \to \infty$, then Y_n is said to have an asymptotic normal distribution with asymptotic mean m and asymptotic variance c^2/n

EX: Ex 4.6.3. $n=40$, $X_i \sim EXP(100)$ X_i = lifetime of an electrical part.

By CLT, $\bar{X}_n \sim N(100, \underbrace{\frac{100^2}{40}}_{250})$

This allows us to answer questions easier than the original

Asymptotic Dist of Central Order Stats

We have the following

- $X_1, X_2, \ldots, X_n$ is a RS from a cont. dist, which is nonzero at X_p and
- X_p is the p^{th} percentile (different than x's above) $0 < p < 1$
- Suppose $\frac{K}{n} \to p$ (with $K = np$ unbounded), Note: $p \neq 0$ or $p \neq 1$

then

The sequence of K^{th} order stats ($X_{K:n}$) is asymptotically normal with mean X_p and asymptotic variance $\frac{c^2}{n}$, where

$$c^2 = \frac{p(1-p)}{[f(X_p)]^2}.$$

In other words, $\frac{X_{K:n} - X_p}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$

or $X_{K:n} \,\dot{\sim}\, N\left(X_p, \frac{c^2}{n}\right)$

Note! This doesn't work for largest or smallest order stat because the p is 0 or 1 for those!

Ex: Let $X \sim PAR(1,1)$

Find the asymptotic dist of sample median.

The CDF of $PAR(1,1)$ is $F(x) = 1 - \frac{1}{1+x}$ $p = 0.5$

$$F(x_{0.5}) = 0.5 = 1 - \frac{1}{1+x_{0.5}} \Longrightarrow \boxed{x_{0.5}=1} \text{ Median is } 1.$$

The pdf $f(x)$ is $f(x) = \frac{1}{(1+x)^2}$, So $f(x_{0.5}) = \frac{1}{(1+1)^2} = \frac{1}{4}$

$$\text{So } c^2 = \frac{0.5(1-0.5)}{[f(x_{0.5})]^2} = \frac{1/4}{[1/4]^2} = 4$$

It follows that as $\frac{k}{n} \longrightarrow p$, $X_{k:n} \,\dot\sim\, N\left(1, \frac{4}{n}\right)$