[7.6] Prop of Stochastic Convergence.

Properties of stochastic Convergence

This is when a sequence of RVs converge to a constant.

Thm: The sequence $Y_1, Y_2, \ldots$ converges stochastically to c iff $\forall \varepsilon > 0$, $\lim_{n\to\infty} P[|Y_n - c| < \varepsilon] = 1$

When this is satisfied, we say the sequence converges in probability

In stat Notation, we say $Y_n \xrightarrow{P} c$

EX: Revist Bernoulli Law of Large Numbers:

Chebychev's says $P[|X-\mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2}$.

Like before, let $X_i \sim$ RS from $BIN(1, p)$ and $\hat{p}_n = \frac{1}{n}\sum X_i$

$$\mu = E(\hat{p}_n) = \frac{1}{n}\sum_1^n E(X_i) = \frac{1}{n}\sum_1^n p = \frac{np}{n} = p$$

Similarly, $V(\hat{p}) = \sigma^2 = \frac{1}{n^2}\sum V(x_i) = \frac{1}{n^2}\,npq = \frac{pq}{n}$

So, by Chebychev's, it follows

$$P[|\hat{p}_n - \overset{\mu}{p}| < \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2} = 1 - \frac{pq}{n\varepsilon^2}$$

Finally, since $\lim\limits_{n\to\infty} P[|\hat{p}_n - p|] \geq \lim\limits_{n\to\infty} 1 - \frac{pq}{n\varepsilon^2} = 1$

then $P[|\hat{p}_n - p| < \epsilon] \longrightarrow 1$ as $n \to \infty$ and $\hat{p}_n \xrightarrow{P} p$

Ex: Law of Large Numbers

suppose $X_i \sim$ RS from a dist with mean μ & finite variance σ^2

It follows that the sample means converge in prob to μ.

Proof:

$$E(\bar{X}) = E\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n}\sum \underbrace{E(X_i)}_{\mu} = \frac{n\mu}{n} = \mu$$

$$V(\bar{X}) = V\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n^2}\sum \underbrace{V(X_i)}_{\sigma^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

It follows that

$$P\left[|\bar{X}-\mu| < \varepsilon\right] \geq 1 - \frac{\sigma^2/n}{\varepsilon^2} = 1 - \frac{\sigma^2}{n\varepsilon^2}$$

Thus, the Limit is

$$\lim_{n\to\infty} P\left[|\bar{X}-\mu| < \varepsilon\right] \geq 1 - \lim_{n\to\infty}\frac{\sigma^2}{n\varepsilon^2} = 1$$

Thus, $\bar{X} \xrightarrow{P} \mu$

Thm 7.6.3.

If $Z_n = \frac{\sqrt{n}(Y_n - m)}{c} \to N(0,1)$

then $\boxed{Y_n \xrightarrow{P} m}$