# [7.7] Additional Limit Theorems.

## Additional Limit Thms

**DEF:** Convergence in Probability

The sequence of random variables $Y_1, Y_2, \ldots Y_n$ is said to converge in probability to $Y$, written $Y_n \xrightarrow{p} Y$, if

$$\lim_{n\to\infty} P[|Y_n - Y| < \epsilon] = 1$$

Note: This is a bit more general than stochastic convergence.

When $Y_n \xrightarrow{d} DEG(c)$, then $Y_n \xrightarrow{p} c$

The "c" is usually a constant. In the case above, Y is a Random variable.

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Note further that convergence in probability is stronger than convergence in distribution:

Thm: If $Y_n \xrightarrow{p} Y$, then $Y_n \xrightarrow{d} Y$. (The converse is NOT true)

(Thm 7.6.3 is a limited converse that is true) (sequence of asymp. normal variables converges in prob. to the asymp. mean)

If $Y_n \xrightarrow{p} c$, then for any function $g(y)$ that is continuous at $c$,

$$g(Y_n) \xrightarrow{p} g(c)$$

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Proof: Because $g(y)$ is continuous at $c$, then

$\forall \epsilon > 0, \delta > 0$ such that

$|y - c| < \delta \Longrightarrow |g(Y_n) - g(c)| < \epsilon$. Then it follows

that $P[|y - c| < \delta] \leqslant P[|g(Y_n) - g(c)| < \epsilon]$

Because $Y_n \xrightarrow{p} c$, then $\lim_{n\to\infty} P[|Y_n - c| < \delta] = 1$

Thus, $\lim_{n\to\infty} P[|Y_n - c| < \delta] = 1 \leqslant P[|g(Y_n) - g(c)| < \epsilon]$

The only way $\lim_{n\to\infty} P[|g(Y_n) - g(c)| < \epsilon] \geqslant 1$

is if $\lim_{n\to\infty} P[|g(Y_n) - g(c)| < \epsilon] = 1$

Thus, $g(Y_n) \xrightarrow{p} g(c)$

Thm: If $X_n$ and $Y_n$ are sequences of RV and $X_n \xrightarrow{P} c$, $Y_n \xrightarrow{P} d$, then

1. $aX_n + bY_n \xrightarrow{P} ac + bd$
2. $X_n Y_n \xrightarrow{P} cd$
3. $X_n/c \xrightarrow{P} 1$, if $c \neq 0$.
4. $1/\sqrt{X_n} \xrightarrow{p} 1/\sqrt{c}$, if $c \neq 0$, $P[X_n \neq 0] = 1 \quad \forall n$
5. $\sqrt{X_n} \xrightarrow{P} \sqrt{c}$, if $P[X_n \geq 0] = 1$ for all n.

EX: Suppose $Y_n \sim BIN(n, p)$

Then

$$\hat{p} = \frac{Y_n}{n} \xrightarrow{P} p \quad \left( \text{Bernoulli Law of Large Num} \right.$$

Thus

$$\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$$

Thm: Slutsky's Thm

If $X_n, Y_n$ are two sequences of RV $\ni$ $X_n \xrightarrow{P} c$ and $Y_n \xrightarrow{d} Y$, then

1. $X_n + Y_n \xrightarrow{d} c + Y$
2. $X_n Y_n \xrightarrow{d} cY$
3. $\dfrac{Y_n}{X_n} \xrightarrow{d} \dfrac{Y}{c}$, $c \neq 0$

EX: Let $X_i \sim BIN(1, p)$ RS of n.

We know

$$\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} Z \sim N(0,1) \quad \text{CLT}$$

and also

Then

$$\frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \xrightarrow{d} Z \sim N(0,1)$$

Thm: If $Y_n \xrightarrow{d} Y$, then for continuous $g(y)$

$$g(Y_n) \xrightarrow{d} g(Y)$$

Note: $g(y)$ does <u>not</u> depend on n.

<u>Thm</u> If $\frac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} Z \sim N(0,1)$ and if $g'(m) \neq 0$, then

$$\frac{\sqrt{n}[g(Y_n) - g(m)]}{|c\, g'(m)|} \xrightarrow{d} Z \sim N(0,1)$$

In other words, $g(Y_n) \dot{\sim} N\left\{g(m), \frac{c^2[g'(m)]^2}{n}\right\}$

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EX: By the CLT,

Let $g(x) = x^2 \Rightarrow g'(x) = 2x$

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1), \text{ then } \overline{X}_n^2 \dot{\sim} N\left(\mu^2, \frac{\sigma^2(2\mu)^2}{n}\right)$$

(or $\overline{X}_n \dot{\sim} N(\mu, \sigma^2/n)$)