[8.3] Sampling Dist.

A statistic is a random variable!

- It has a distribution
- We call this the sampling distribution
  or derived distribution.

Linear Combinations of Normal Variables

Thm: If $X_i \sim N(\mu_i, \sigma_i^2); i = 1, ..., n$ denote independent RV's, then

$$Y = \sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$ Proof: Use MGF's

Corollary: If $X_1, X_2, \ldots, X_n$ denotes a RS from $N(\mu, \sigma^2)$, then $\overline{X} \sim N(\mu, \sigma^2/n)$.

Compare to CLT!

which is equivalent to $\overline{X} \dot{\sim} N(\mu, \sigma^2/n)$!

If $X_1, X_2, \ldots, X_n$ denotes a RS with mean μ variance σ^2, then $\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \xrightarrow{d} N(0, 1)$,

The Chi-Squared Distribution χ^2

prounouce "Chi" as in kite

If $Y \sim GAM(2, \nu/2)$, where $\nu=$ degrees of freedom, then
Y is said to follow a χ^2 distribution with degrees of freedom ν

In stats notation, $Y \sim \chi^2(\nu)$

Thm: If $Y \sim \chi^2(\nu)$, then

$$M_Y(t) = (1-2t)^{-\nu/2} \qquad \text{and} \qquad E[Y] = \nu$$

$$E[Y^r] = \frac{2^r \Gamma\left(\frac{\nu}{2}+r\right)}{\Gamma\left(\frac{\nu}{2}\right)} \qquad \mathrm{Var}[Y] = 2\nu$$

Thm: If $X \sim GAM(\theta, \kappa)$, then $Y = \dfrac{2X}{\theta} \sim \chi^2(2\kappa)$.

Proof: Use MGFs Note: $M_X(t) = (1-\theta t)^{-\kappa}$

$$M_Y(t) = E[e^{tY}] = E[e^{t\frac{2X}{\theta}}] = E[e^{\frac{2t}{\theta}X}] = M_X\left(\frac{2t}{\theta}\right) = \left(1-\theta\left(\frac{2t}{\theta}\right)\right)^{-\kappa} = (1-2t)^{-2\kappa/2}$$

That's the MGF for the $\chi^2(2\kappa)$ dist!

We can use a Chi-square table to compute the CDF for ANY Gamma dist.
You can consider it like a "Standard Gamma Dist"

In particular, if $X \sim GAM(\theta, \kappa)$, then $F_X(x) = F_Y\left(\frac{2X}{\theta}; 2\kappa\right)$, where $Y \sim \chi^2(2\kappa)$

EX: X=time in years until failure of a component $\quad X \sim GAM(\theta = 3, \kappa = 2)$

Find the 10th percentile

$$P[X \leqslant x_{0.10}] = F_Y\left(\frac{2x_{0.10}}{\theta}\right) = 0.10$$

So $\frac{2x_{0.10}}{\theta} = \chi^2_{0.10}(2\kappa) \leftarrow$ Look up in the χ^2 table.

$$\frac{2x_{0.10}}{3} = \chi^2_{0.10}(4)$$

Thus, $x_{0.10} = \frac{3}{2}\chi^2_{0.10}(4) = \frac{3}{2}(1.06) = 1.59$

Try it for problem 37 or 38 in chapter 3 from your textbook.

In that one, X is GAM(5,4)
or X is GAM(1,3)

You can use it to find prob. as well as percentiles.

Thm: the sum of independent χ^2 RVs Thm: If $Y_i \sim \chi^2(\nu_i)$ are i.d., then $V = \sum_{i=1}^{n} Y_i \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right)$

Thm: the Square of Standard Normal is a Chi-square with df=1

If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$.

Corollary 8.3.2

If $X_1, X_2, \ldots, X_n$ denotes a random sample from $N(\mu, \sigma^2)$, then

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n) \quad \text{and} \quad \frac{n(\overline{X} - \mu)}{\sigma^2} \sim \mathrm{N}(0,1)$$

Proof:

Since $\frac{X_i - \mu}{\sigma} \sim N(0,1)$, then $\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(1) \Longrightarrow$ Thus, $\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2\left(\sum_{i=1}^{n} 1\right) = \chi^2(n)$

Since $\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, then $\left(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2(1)$

What is the distribution of $S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$?

This is difficult to know since $X_i - \overline{X}$ is not independent of $X_j - \overline{X}$. Note $\sum_{i=1}^{n} X_i - \overline{X} = 0$ always!

Thm: If $X_1, X_2, \ldots, X_n$ denotes a RS from $N(\mu, \sigma^2)$, then

1. $\overline{X}$ and the terms $X_i - \overline{X}; i = 1, \ldots, n$ are independent
2. $\overline{X}$ and S^2 are independent.
3. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$