

## [8.5] Large Sample Approx.

Thm:  $Y_v \sim \chi^2(v)$ , then

$$Z_v = \frac{Y_v - v}{\sqrt{2v}} \xrightarrow{d} Z \sim N(0,1)$$

**Proof:** The Central Limit Theorem states:

$$\frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1)$$

where  $\mu = E(X_i)$  and  $\sigma^2 = \text{Var}(X_i)$ .

If  $X_i \sim \chi^2(1)$ , then

$$Y_v = \sum_{i=1}^v X_i \sim \chi^2(v)$$

$$\mu = E(X_i) = 1$$

$$\sigma^2 = \text{Var}(X_i) = 2 \Rightarrow \sigma = \sqrt{2}$$

Thus

$$\frac{Y_v - v}{\sqrt{2v}} = \frac{\sum X_i - v(1)}{\sqrt{v} \cdot \sqrt{2}} \xrightarrow{d} N(0,1)$$

$$\gamma = P\left[\chi^2_V \leq \underbrace{\chi^2_{\gamma}(V)}_{\text{percentile}}\right] = \Phi\left(\frac{\underbrace{\chi^2_{\gamma}(V) - V}_{z_{\gamma}}}{\sqrt{2V}}\right)$$

$$\Rightarrow z_{\gamma} = \frac{\chi^2_{\gamma}(V) - V}{\sqrt{2V}}$$

$$\boxed{\chi^2_{\gamma}(V) \doteq V + z_{\gamma}\sqrt{2V}}$$

Ex:  $V = 30 \quad \gamma = .95$

$$\chi^2_{.95}(30) \doteq 30 + 1.645(\sqrt{60})$$

$$= 42.74$$

$$\chi^2_{.95}(30) = 43.77$$

A more accurate approx

$$\chi^2_{\gamma}(V) \doteq V \left[ 1 - \frac{2}{9V} + z_{\gamma} \sqrt{\frac{2}{9V}} \right]^3$$

$$\chi^2_{.95}(30) = 30 \left[ 1 - \frac{2}{9(30)} + 1.645 \sqrt{\frac{2}{9(30)}} \right]^3$$

$$= 43.768.$$

Wilson-Hilferty Approx.

Don't forget the 3!

ex:  $V_n = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$

So, by the theorem above (on first page), it follows that

$$\frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} z \sim N(0, 1).$$

So,

$$\begin{aligned} \frac{\frac{(n-1)S_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} &= \frac{\frac{(n-1)S^2}{\sigma^2} - \frac{(n-1)\sigma^2}{\sigma^2}}{\sqrt{2(n-1)}} = \frac{\frac{(n-1)}{\sigma^2} (S^2 - \sigma^2)}{\sqrt{2} \sqrt{n-1}} \\ &= \frac{\sqrt{n-1} (S_n^2 - \sigma^2)}{\sqrt{2} \sigma^2} \xrightarrow{d} z \sim N(0, 1) \end{aligned}$$

Annotations: Red arrows point from the first fraction to the second. A red arrow labeled "factor!" points to the circled term  $\frac{(n-1)}{\sigma^2} (S^2 - \sigma^2)$ . A cyan arrow points from the  $\sqrt{2} \sqrt{n-1}$  term to the denominator of the final fraction.

So, for large  $n$ ,

$$S_n^2 \sim N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

Using Thm 7.7.6:

$$\text{Let } g(y) = \sqrt{y} \quad g'(y) = \frac{1}{2\sqrt{y}}$$

$$\text{So } g'(\sigma^2) = \frac{1}{2\sqrt{\sigma^2}} = \frac{1}{2\sigma}$$

$$\begin{aligned} \text{Then: } \frac{S - \sigma}{\sqrt{\frac{2\sigma^4}{n-1} \left(\frac{1}{2\sigma}\right)^2}} &= \frac{S - \sigma}{\sqrt{\frac{2\sigma^4}{n-1} \left(\frac{1}{4\sigma^2}\right)}} \\ &= \frac{S - \sigma}{\sqrt{\frac{\sigma^2}{2(n-1)}}} \xrightarrow{d} N(0, 1) \end{aligned}$$

$$\text{Thus, } S \sim N\left(\sigma, \frac{\sigma^2}{2(n-1)}\right)$$

EX:

$t$  has a limiting normal dist.

$$E\left[\frac{\chi_v^2}{v}\right] = \frac{1}{v} E(\chi_v^2) = \frac{v}{v} = 1$$

$$V\left[\frac{\chi_v^2}{v}\right] = \frac{1}{v^2} V(\chi_v^2) = \frac{1}{v^2} 2v = \frac{2}{v}$$

Chebyshev's Inequality.

$$P\left[\left|\frac{\chi_v^2}{v} - 1\right| < \varepsilon\right] \geq 1 - \frac{2}{v\varepsilon^2} \xrightarrow{v \rightarrow \infty} 1$$

$$\frac{\chi_v^2}{v} \xrightarrow{P} 1$$

$$T_v = \frac{z}{\sqrt{\frac{\chi_v^2}{v}}}$$

$$\sqrt{\frac{\chi_v^2}{v}} \xrightarrow{P} \sqrt{1} = 1$$

$$T_v = \frac{z}{\sqrt{\frac{\chi_v^2}{v}}} \xrightarrow{d} \frac{z}{1} \sim N(0,1)$$

Slutsky's Thm, pt 3.

$$a_n X_n \rightarrow aX$$

$$a_n \xrightarrow{P} a \quad X_n \xrightarrow{d} X$$