[8.5] Large Sample Approx.

Thm: $Y_\nu \sim \chi^2(\nu)$, then

$$Z_\nu = \frac{Y_\nu - \nu}{\sqrt{2\nu}} \xrightarrow{d} Z \sim N(0,1)$$

Proof: The Central Limit Theorem states:

$$\frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{d} N(0,1)$$

where $\mu = E(X_i)$ and $\sigma^2 = \mathrm{Var}(X_i)$.

If $X_i \sim \chi^2(1)$, then

$$Y_\nu = \sum_{i=1}^{\nu} X_i \sim \chi^2(\nu)$$

$$\mu = E(X_i) = 1$$

$$\sigma^2 = V(X_i) = 2 \Rightarrow \sigma = \sqrt{2}$$

Thus

$$\frac{Y_\nu - \nu}{\sqrt{2\nu}} = \frac{\sum X_i - \nu(1)}{\sqrt{\nu}\,\sqrt{2}} \xrightarrow{d} N(0,1)$$

$$\gamma = P[Y_\nu \le \underbrace{\chi^2_\gamma(\nu)}_{\text{percentile}}] \doteq \Phi\left(\underbrace{\frac{\chi^2_\gamma(\nu)-\nu}{\sqrt{2\nu}}}_{z_\gamma}\right)$$

$$\Rightarrow \quad z_\gamma \doteq \frac{\chi^2_\gamma(\nu)-\nu}{\sqrt{2\nu}}$$

$$\boxed{\chi^2_\gamma(\nu) \doteq \nu + z_\gamma\sqrt{2\nu}}$$

Ex: $\nu = 30 \quad \gamma = .95$

$$\chi^2_{.95}(30) \doteq 30 + 1.645(\sqrt{60})$$

$$= 42.74$$

$$\chi^2_{.95}(30) = 43.77$$

A more accurate approx.

$$\chi^2_\gamma(\nu) \doteq \nu\left[1 - \frac{2}{9\nu} + z_\gamma\sqrt{\frac{2}{9\nu}}\right]^3$$

$$\chi^2_{.95}(30) = 30\left[1 - \frac{2}{9(30)} + 1.645\sqrt{\frac{2}{9(30)}}\right]^3$$

$$= 43.768.$$

Wilson-Hilferty Approx.

Don't forget the 3!

ex: $V_n = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$

$\boxed{\text{So, by the theorem above (on first page), it follows that}}$

$$\frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0,1).$$

So,

$$\frac{\frac{(n-1)S_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\frac{(n-1)S^2}{\sigma^2} - \frac{(n-1)\sigma^2}{\sigma^2}}{\sqrt{2(n-1)}} = \frac{\frac{(n-1)}{\sigma^2}(S^2-\sigma^2)}{\sqrt{2}\sqrt{n-1}}$$

factor!

$$= \frac{\sqrt{n-1}\,(S_n^2-\sigma^2)}{\sqrt{2}\,\sigma^2} \xrightarrow{d} Z \sim N(0,1)$$

So, for large n,

$$S_n^2 \sim N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

Using Thm 7.7.6:

Let $g(y) = \sqrt{y}$ $\quad g'(y) = \frac{1}{2\sqrt{y}}$

So $g'(\sigma^2) = \frac{1}{2\sqrt{\sigma^2}} = \frac{1}{2\sigma}$

Then: $$\frac{S-\sigma}{\sqrt{\frac{2\sigma^4}{n-1}\left(\frac{1}{2\sigma}\right)^2}} = \frac{S-\sigma}{\sqrt{\frac{2\sigma^4}{n-1}\left(\frac{1}{4\sigma^2}\right)}}$$

$$= \frac{S-\sigma}{\sqrt{\frac{\sigma^2}{2(n-1)}}} \xrightarrow{d} N(0,1)$$

Thus, $S \mathrel{\dot\sim} N\left(\sigma, \frac{\sigma^2}{2(n-1)}\right)$

Ex: $\boxed{t \text{ has a limiting normal dist.}}$

$$E\left[\frac{\chi^2_\nu}{\nu}\right] = \frac{1}{\nu}E(\chi^2_\nu) = \frac{\nu}{\nu} = 1$$

$$V\left[\frac{\chi^2_\nu}{\nu}\right] = \frac{1}{\nu^2}V(\chi^2_\nu) = \frac{1}{\nu^2}2\nu = \frac{2}{\nu}$$

Chebychev's Inequality.

$$P\left[\left|\frac{\chi^2_\nu}{\nu} - 1\right| < \varepsilon\right] \geq 1 - \frac{2}{\nu\varepsilon^2} \xrightarrow{\nu\to\infty} 1$$

$$\frac{\chi^2_\nu}{\nu} \xrightarrow{P} 1$$

$$T_\nu = \frac{Z}{\sqrt{\frac{\chi^2_\nu}{\nu}}}$$

$$\sqrt{\frac{\chi^2_\nu}{\nu}} \xrightarrow{P} \sqrt{1} = 1$$

$$T_\nu = \frac{Z}{\sqrt{\frac{\chi^2_\nu}{\nu}}} \xrightarrow{d} \frac{Z}{1} \sim N(0,1)$$

Slutsky's Thm, pt 3.

$$a_n X_n \to aX$$

$$a_n \xrightarrow{P} a \quad X_n \xrightarrow{d} X$$