[9.3] Criteria for evaluating estimators

Uniformly Minimum Variance Unbiased Estimators (UMVUE)

DEF: Let $X_1, X_2, \ldots, X_n$ be a RS from $f(x;\theta)$

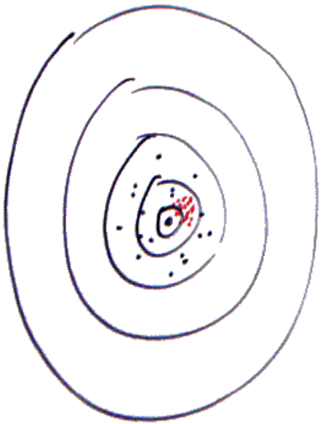
An estimator T^* of $\tau(\theta)$ is called the UMVUE of $\tau(\theta)$ if.

1. T^* is unbiased for $\tau(\theta)$
2. For any other unbiased est T of $\tau(\theta)$

$$Var(T^*) \le Var(T) \text{ for all } \theta \in \Omega$$

If T is an unbiased est of $\tau(\theta)$, then the Cramer-Rao lower bound (CRLB) is

$$Var(T) \geq \frac{[\tau'(\theta)]^2}{E\left\{\left[\frac{\partial}{\partial\theta} \ln(f(x;\theta))\right]^2\right\}}$$



DEF: If T is an est of $\tau(\theta)$, then the bias is given by

$$b(T) = E(T) - \tau(\theta)$$
$$= E\left(T - \tau(\theta)\right)$$

An unbiased est, T, for $\tau(\theta)$ has the prop:

$$E(T) = \tau(\theta)$$

$$b(T) = E(T) - \tau(\theta) = 0.$$

the mean-squared error (MSE) is given by

$$MSE(T) = E[T - \tau(\theta)]^2$$

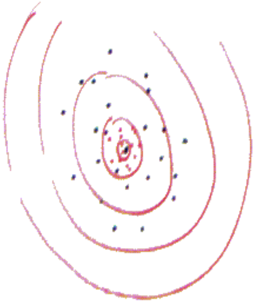
Thm: If T is an est of $\tau(\theta)$, then

$$MSE(T) = var(T) + [b(T)]^2$$

$$\sigma^2_{mle} = \frac{n-1}{n} S^2$$

$$\lim_{n\to\infty} \sigma^2_{mle} = S^2$$

One idea for choosing an estimator is to choose the one that tends to be closest or "most concentrated" around the true value.



It might be reasonable to say that T_1 is better than T_2 if

$$P[\tau(\theta)-\varepsilon < T_1 < \tau(\theta)+\varepsilon] \geq P[\tau(\theta)-\varepsilon < T_2 < \tau(\theta)+\varepsilon]$$

for all $\varepsilon > 0$,

Note: By chebychev's

$$P[\tau(\theta)-\varepsilon < T < \tau(\theta)+\varepsilon] = P[-\varepsilon < T-\tau(\theta) < \varepsilon] = P[|T-\tau(\theta)| < \varepsilon]$$

$$\geq 1-\frac{Var(T-\tau(\theta))}{\varepsilon^2} = 1-\frac{Var(T)}{\varepsilon^2}$$

Our goal: Pick T^* such that $Var(T^*) \leq Var(T)$!

Note: If proper differentiability cond. hold, then it can be shown

$$E\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2 = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]$$

Ex: RS from $X_i \sim EXP(\theta)$.

$$\ln f(x;\theta) = -\frac{x}{\theta} - \ln\theta$$

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = \frac{x}{\theta^2} - \frac{1}{\theta} = \frac{x-\theta}{\theta^2}$$

$$E\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right] = E\left[\frac{(x-\theta)^2}{\theta^4}\right] = \frac{1}{\theta^4}E(x-\theta)^2$$

$$= \frac{Var(x)}{\theta^4} = \frac{\theta^2}{\theta^4} = \frac{1}{\theta^2}$$

$T^* = \bar{X}$

CRLB is: $E(x)=\theta \Rightarrow E(\bar{x}) = \mu = \theta$ $\quad \tau(\theta) = \theta$

$$Var(T) = \frac{[\tau'(\theta)]^2}{nE\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2} = \frac{1}{n\cdot\frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

$\bar{x}$ is the UMVUE of θ.

$E(\bar{x}) = \mu$

$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$